Chapter 11

Circles

11A Lines and Arcs in Circles
11-1 Lines That Intersect Circles
11-2 Arcs and Chords
11-3 Sector Area and Arc Length

11B Angles and Segments in Circles
11-4 Inscribed Angles
Lab Explore Angle Relationships in Circles
11-5 Angle Relationships in Circles
Lab Explore Segment Relationships in Circles
11-6 Segment Relationships in Circles
11-7 Circles in the Coordinate Plane
Ext Polar Coordinates

Elegant Eggs!
Constructing egg shapes from curves of different radii is an ancient art form that goes back to prehistoric times.
Vocabulary
Match each term on the left with a definition on the right.

1. radius
   A. the distance around a circle
2. pi
   B. the locus of points in a plane that are a fixed distance from a given point
3. circle
   C. a segment with one endpoint on a circle and one endpoint at the center of the circle
4. circumference
   D. the point at the center of a circle
   E. the ratio of a circle's circumference to its diameter

Tables and Charts
The table shows the number of students in each grade level at Middletown High School. Find each of the following.

5. the percentage of students who are freshman
6. the percentage of students who are juniors
7. the percentage of students who are sophomores or juniors

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freshman</td>
<td>192</td>
</tr>
<tr>
<td>Sophomore</td>
<td>208</td>
</tr>
<tr>
<td>Junior</td>
<td>216</td>
</tr>
<tr>
<td>Senior</td>
<td>184</td>
</tr>
</tbody>
</table>

Circle Graphs
The circle graph shows the age distribution of residents of Mesa, Arizona, according to the 2000 census. The population of the city is 400,000.

8. How many residents are between the ages of 18 and 24?
9. How many residents are under the age of 18?
10. What percentage of the residents are over the age of 45?
11. How many residents are over the age of 45?

Solve Equations with Variables on Both Sides
Solve each equation.

12. \(11y - 8 = 8y + 1\)
13. \(12x + 32 = 10 + x\)
14. \(z + 30 = 10z - 15\)
15. \(4y + 18 = 10y + 15\)
16. \(-2x - 16 = x + 6\)
17. \(-2x - 11 = -3x - 1\)

Solve Quadratic Equations
Solve each equation.

18. \(17 = x^2 - 32\)
19. \(2 + y^2 = 18\)
20. \(4x^2 + 12 = 7x^2\)
21. \(188 - 6x^2 = 38\)
Previously, you
- used the fundamental vocabulary of circles.
- developed and applied formulas for the area and circumference of circles.
- used circles to solve problems.

You will study
- solving problems involving circles.
- finding lengths, angle measures, and areas associated with circles.
- applying circle theorems to solve a wide range of problems.

You can use the skills learned in this chapter
- in Algebra 2 and Precalculus.
- in other classes, such as in Biology when you explore cells, in Physics when you study the laws of motion and kinematic principles, and in Art when you create images.
- to calculate distances, interpret information in newspaper and magazine charts, and make designs.

Key Vocabulary/Vocabulario

<table>
<thead>
<tr>
<th>English</th>
<th>Spanish</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc</td>
<td>arco</td>
</tr>
<tr>
<td>arc length</td>
<td>longitud de arco</td>
</tr>
<tr>
<td>central angle</td>
<td>ángulo central</td>
</tr>
<tr>
<td>chord</td>
<td>cuerda</td>
</tr>
<tr>
<td>secant</td>
<td>secante</td>
</tr>
<tr>
<td>sector of a circle</td>
<td>sector de un círculo</td>
</tr>
<tr>
<td>segment of a circle</td>
<td>segmento de un círculo</td>
</tr>
<tr>
<td>semicircle</td>
<td>semicírculo</td>
</tr>
<tr>
<td>tangent of a circle</td>
<td>tangente de un círculo</td>
</tr>
</tbody>
</table>

Vocabulary Connections

To become familiar with some of the vocabulary terms in the chapter, answer the following questions. You may refer to the chapter, the glossary, or a dictionary if you like.

1. The word **semicircle** begins with the prefix **semi-**. List some other words that begin with **semi-**. What do all of these words have in common?

2. The word **central** means “located at the center.” How can you use this definition to understand the term **central angle** of a circle?

3. The word **tangent** comes from the Latin word **tangere**, which means “to touch.” What does this tell you about a line that is a tangent to a circle?

4. The word **secant** comes from the Latin word **secare**, which means “to cut.” What does this tell you about a line that is a secant to a circle?
Reading Strategy: Read to Solve Problems

A word problem may be overwhelming at first. Once you identify the important parts of the problem and translate the words into math language, you will find that the problem is similar to others you have solved.

**Reading Tips:**
- ✔ Read each phrase slowly. Write down what the words mean as you read them.
- ✔ Draw a diagram. Label the diagram so it makes sense to you.
- ✔ Read the problem again before finding your solution.
- ✔ Translate the words or phrases into math language.
- ✔ Highlight what is being asked.

**From Lesson 10-3:** Use the Reading Tips to help you understand this problem.

1. After a day hike, a group of hikers set up a camp 3 km east and 7 km north of the starting point. The elevation of the camp is 0.6 km higher than the starting point. What is the distance from the camp to the starting point?

Use the Distance Formula to find the distance between the camp and the starting point.

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]

\[
= \sqrt{(3 - 0)^2 + (7 - 0)^2 + (0.6 - 0)^2} \approx 7.6 \text{ km}
\]

**Try This**

For the following problem, apply the following reading tips. Do not solve.
- Identify key words.
- Translate each phrase into math language.
- Draw a diagram to represent the problem.

1. The height of a cylinder is 4 ft, and the diameter is 9 ft. What effect does doubling each measure have on the volume?
**Objectives**
Identify tangents, secants, and chords.
Use properties of tangents to solve problems.

**Vocabulary**
- Interior of a circle
- Exterior of a circle
- Chord
- Secant
- Tangent of a circle
- Point of tangency
- Concentric circles
- Tangent circles
- Common tangent

**Why learn this?**
You can use circle theorems to solve problems about Earth. (See Example 3.)

This photograph was taken 216 miles above Earth. From this altitude, it is easy to see the curvature of the horizon. Facts about circles can help us understand details about Earth.

Recall that a circle is the set of all points in a plane that are equidistant from a given point, called the center of the circle. A circle with center \( C \) is called circle \( C \), or \( \odot C \).

The **interior of a circle** is the set of all points inside the circle. The **exterior of a circle** is the set of all points outside the circle.

**Lines and Segments That Intersect Circles**

<table>
<thead>
<tr>
<th>TERM</th>
<th>DIAGRAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>A chord</td>
<td><img src="image" alt="Chord Diagram" /></td>
</tr>
<tr>
<td>A secant</td>
<td><img src="image" alt="Secant Diagram" /></td>
</tr>
<tr>
<td>A tangent</td>
<td><img src="image" alt="Tangent Diagram" /></td>
</tr>
</tbody>
</table>

The point where the tangent and a circle intersect is called the **point of tangency**.

**Example 1**
Identifying Lines and Segments That Intersect Circles

Identify each line or segment that intersects \( \odot A \).

- Chords: \( EF \) and \( BC \)
- Tangent: \( \ell \)
- Radii: \( AC \) and \( AB \)
- Secant: \( \overrightarrow{EF} \)
- Diameter: \( BC \)
1. Identify each line or segment that intersects $\odot P$.

Remember that the terms *radius* and *diameter* may refer to line segments, or to the lengths of segments.

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**Pairs of Circles**

<table>
<thead>
<tr>
<th>TERM</th>
<th>DIAGRAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two circles are <strong>congruent circles</strong> if and only if they have congruent radii.</td>
<td><img src="image1" alt="Diagram of congruent circles" /></td>
</tr>
<tr>
<td><strong>Concentric circles</strong> are coplanar circles with the same center.</td>
<td><img src="image2" alt="Diagram of concentric circles" /></td>
</tr>
<tr>
<td>Two coplanar circles that intersect at exactly one point are called <strong>tangent circles</strong>.</td>
<td><img src="image3" alt="Diagram of tangent circles" /></td>
</tr>
</tbody>
</table>

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**Example 2**

**Identifying Tangents of Circles**

Find the length of each radius. Identify the point of tangency and write the equation of the tangent line at this point.

- **Center** of $\odot A$: $(-1, 0)$, **Pt. on $\odot$** is $(3, 0)$, **Dist. between the 2 pts.** is 4.
- **Center** of $\odot B$: $(1, 0)$, **Pt. on $\odot$** is $(3, 0)$, **Dist. between the 2 pts.** is 2.

<table>
<thead>
<tr>
<th>point of tangency: $(3, 0)$</th>
<th><strong>Pt. where the $\odot$s and tangent line intersect</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>equation of tangent line: $x = 3$</td>
<td><strong>Vert. line through $(3, 0)$</strong></td>
</tr>
</tbody>
</table>

---

2. Find the length of each radius. Identify the point of tangency and write the equation of the tangent line at this point.
A common tangent is a line that is tangent to two circles.

Lines \( \ell \) and \( m \) are common external tangents to \( \odot A \) and \( \odot B \).

Lines \( p \) and \( q \) are common internal tangents to \( \odot A \) and \( \odot B \).

**Construction** Tangent to a Circle at a Point

1. Draw \( \odot P \). Locate a point on the circle and label it \( Q \).
2. Draw \( \overline{PQ} \).
3. Construct the perpendicular \( \ell \) to \( \overline{PQ} \) at \( Q \). This line is tangent to \( \odot P \) at \( Q \).

Notice that in the construction, the tangent line is perpendicular to the radius at the point of tangency. This fact is the basis for the following theorems.

**Theorems**

<table>
<thead>
<tr>
<th>THEOREM</th>
<th>HYPOTHESIS</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>11-1-1</strong></td>
<td>If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency. (line tangent to ( \odot ) ( \to ) line ( \perp ) to radius)</td>
<td>( \ell \perp \overline{AB} )</td>
</tr>
<tr>
<td><strong>11-1-2</strong></td>
<td>If a line is perpendicular to a radius of a circle at a point on the circle, then the line is tangent to the circle. (line ( \perp ) to radius ( \to ) line tangent to ( \odot ))</td>
<td>( m ) is ( \perp ) to ( \overline{CD} ) at ( D )</td>
</tr>
</tbody>
</table>

You will prove Theorems 11-1-1 and 11-1-2 in Exercises 28 and 29.
**Problem Solving Application**

The summit of Mount Everest is approximately 29,000 ft above sea level. What is the distance from the summit to the horizon to the nearest mile?

1. **Understand the Problem**

   The answer will be the length of an imaginary segment from the summit of Mount Everest to Earth’s horizon.

2. **Make a Plan**

   Draw a sketch. Let $C$ be the center of Earth, $E$ be the summit of Mount Everest, and $H$ be a point on the horizon. You need to find the length of $EH$, which is tangent to $\odot C$ at $H$. By Theorem 11-1-1, $EH \perp CH$. So $\triangle CHE$ is a right triangle.

3. **Solve**

   \[
   ED = 29,000 \text{ ft} \\
   = \frac{29,000}{5280} \approx 5.49 \text{ mi} \\
   EC = CD + ED \\
   = 4000 + 5.49 = 4005.49 \text{ mi} \\
   EC^2 = EH^2 + CH^2 \\
   4005.49^2 = EH^2 + 4000^2 \\
   43,950.14 \approx EH^2 \\
   210 \text{ mi} \approx EH
   \]

4. **Look Back**

   The problem asks for the distance to the nearest mile. Check if your answer is reasonable by using the Pythagorean Theorem. Is $210^2 + 4000^2 \approx 4005^2$? Yes, $16,044,100 \approx 16,040,025$.

3. Kilimanjaro, the tallest mountain in Africa, is 19,340 ft tall. What is the distance from the summit of Kilimanjaro to the horizon to the nearest mile?

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### Theorem 11-1-3

<table>
<thead>
<tr>
<th>THEOREM</th>
<th>HYPOTHESIS</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>If two segments are tangent to a circle from the same external point, then the segments are congruent. (2 segs. tangent to $\odot$ from same ext. pt. $\rightarrow$ segs. \cong)</td>
<td>$\overline{AB}$ and $\overline{AC}$ are tangent to $\odot P$.</td>
<td>$\overline{AB} \cong \overline{AC}$</td>
</tr>
</tbody>
</table>

You will prove Theorem 11-1-3 in Exercise 30.
You can use Theorem 11-1-3 to find the length of segments drawn tangent to a circle from an exterior point.

**Example 4**

Using Properties of Tangents

\( \overline{DE} \) and \( \overline{DF} \) are tangent to \( \odot C \). Find \( DF \).

\[
\begin{align*}
DE &= DF & \text{2 segs. tangent to } \odot \text{ from same ext. pt. } \rightarrow \text{ segs. } \cong. \\
5y - 28 &= 3y & \text{Substitute } 5y - 28 \text{ for } DE \text{ and } 3y \text{ for } DF. \\
2y - 28 &= 0 & \text{Subtract } 3y \text{ from both sides.} \\
2y &= 28 & \text{Add } 28 \text{ to both sides.} \\
y &= 14 & \text{Divide both sides by } 2. \\
DF &= 3(14) & \text{Substitute } 14 \text{ for } y. \\
&= 42 & \text{Simplify.}
\end{align*}
\]

\( \overline{RS} \) and \( \overline{RT} \) are tangent to \( \odot Q \). Find \( RS \).

4a.

4b.

**Think and Discuss**

1. Consider \( \odot A \) and \( \odot B \). How many different lines are common tangents to both circles? Copy the circles and sketch the common external and common internal tangent lines.

2. Is it possible for a line to be tangent to two concentric circles? Explain your answer.

3. Given \( \odot P \), is the center \( P \) a part of the circle? Explain your answer.

4. In the figure, \( \overline{RQ} \) is tangent to \( \odot P \) at \( Q \). Explain how you can find \( m\angle PRQ \).

5. **Get Organized** Copy and complete the graphic organizer below. In each box, write a definition and draw a sketch.

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**Know it! Note**

750 Chapter 11 Circles
GUIDED PRACTICE

**Vocabulary** Apply the vocabulary from this lesson to answer each question.

1. A ___?___ is a line in the plane of a circle that intersects the circle at two points. (secant or tangent)

2. Coplanar circles that have the same center are called ___?___. (concentric or congruent)

3. \(\odot Q\) and \(\odot R\) both have a radius of 3 cm. Therefore the circles are ___?____. (concentric or congruent)

**Identify each line or segment that intersects each circle.**

4. 

5. 

**Multi-Step** Find the length of each radius. Identify the point of tangency and write the equation of the tangent line at this point.

6. 

7. 

**Space Exploration** The International Space Station orbits Earth at an altitude of 240 mi. What is the distance from the space station to Earth's horizon to the nearest mile?

8. 

**The segments in each figure are tangent to the circle. Find each length.**

9. \(JK\)

10. \(ST\)
Olympus Mons, located on Mars, is the tallest known volcano in the solar system.

**Astronomy**

Olympus Mons’s peak rises 25 km above the surface of the planet Mars. The diameter of Mars is approximately 6794 km. What is the distance from the peak of Olympus Mons to the horizon to the nearest kilometer?

The segments in each figure are tangent to the circle. Find each length.

**Multi-Step** Find the length of each radius. Identify the point of tangency and write the equation of the tangent line at this point.

Tell whether each statement is sometimes, always, or never true.

18. Two circles with the same center are congruent.
19. A tangent to a circle intersects the circle at two points.
20. Tangent circles have the same center.
21. A tangent to a circle will form a right angle with a radius that is drawn to the point of tangency.
22. A chord of a circle is a diameter.

**Graphic Design** Use the following diagram for Exercises 23–25.

The peace symbol was designed in 1958 by Gerald Holtom, a professional artist and designer. Identify the following.

23. diameter
24. radii
25. chord
In each diagram, \( \overline{PR} \) and \( \overline{PS} \) are tangent to \( \odot Q \). Find each angle measure.

26. \( m\angle Q \)

27. \( m\angle P \)

28. Complete this indirect proof of Theorem 11-1-1.

Given: \( \ell \) is tangent to \( \odot A \) at point \( B \).

Prove: \( \ell \perp AB \)

Proof: Assume that \( \ell \) is not \( \perp AB \). Then it is possible to draw \( AC \) such that \( AC \perp \ell \). If this is true, then \( \triangle ACB \) is a right triangle. \( AC < AB \) because a. \_. Since \( \ell \) is a tangent line, it can only intersect \( \odot A \) at b. \_. and \( C \) must be in the exterior of \( \odot A \). That means that \( AC > AB \) since \( AB \) is a c. \_. This contradicts the fact that \( AC < AB \). Thus the assumption is false, and d. \_.


Given: \( m \perp CD \)

Prove: \( m \) is tangent to \( \odot C \).

(Hint: Choose a point on \( m \). Then use the Pythagorean Theorem to prove that if the point is not \( D \), then it is not on the circle.)

30. Prove Theorem 11-1-3.

Given: \( \overline{AB} \) and \( \overline{AC} \) are tangent to \( \odot P \).

Prove: \( \overline{AB} \cong \overline{AC} \)

Plan: Draw auxiliary segments \( \overline{PA} \), \( \overline{PB} \), and \( \overline{PC} \). Show that the triangles formed are congruent. Then use CPCTC.

**Algebra** Assume the segments that appear to be tangent are tangent.

Find each length.

31. \( ST \)

32. \( DE \)

33. \( JL \)

34. \( \odot M \) has center \( M(2, 2) \) and radius 3. \( \odot N \) has center \( N(-3, 2) \) and is tangent to \( \odot M \). Find the coordinates of the possible points of tangency of the two circles.

35. This problem will prepare you for the Multi-Step Test Prep on page 770.

The diagram shows the gears of a bicycle.

\( AD = 5 \) in., and \( BC = 3 \) in. \( CD \), the length of the chain between the gears, is 17 in.

a. What type of quadrilateral is \( BCDE \)? Why?

b. Find \( BE \) and \( AE \).

c. What is \( AB \) to the nearest tenth of an inch?
36. **Critical Thinking** Given a circle with diameter $BC$, is it possible to draw tangents to $B$ and $C$ from an external point $X$? If so, make a sketch. If not, explain why it is not possible.

37. **Write About It** $PR$ and $PS$ are tangent to $\odot Q$. Explain why $\angle P$ and $\angle Q$ are supplementary.

38. $AB$ and $AC$ are tangent to $\odot D$. Which of these is closest to $AD$?
   - A: 9.5 cm
   - B: 10 cm
   - C: 10.4 cm
   - D: 13 cm

39. $\odot P$ has center $P(3, -2)$ and radius 2. Which of these lines is tangent to $\odot P$?
   - F: $x = 0$
   - G: $y = -4$
   - H: $y = -2$
   - I: $x = 4$

40. $\odot A$ has radius 5, $\odot B$ has radius 6. What is the ratio of the area of $\odot A$ to that of $\odot B$?
   - A: $\frac{125}{216}$
   - B: $\frac{25}{36}$
   - C: $\frac{5}{6}$
   - D: $\frac{36}{25}$

41. **Given:** $\odot G$ with $GH \perp JK$
   **Prove:** $JH \cong KH$

42. **Multi-Step** $\odot A$ has radius 5, $\odot B$ has radius 2, and $CD$ is a common tangent. What is $AB$? (Hint: Draw a perpendicular segment from $B$ to $E$, a point on $AC$.)

43. **Manufacturing** A company builds metal stands for bicycle wheels. A new design calls for a V-shaped stand that will hold wheels with a 13 in. radius. The sides of the stand form a $70^\circ$ angle. To the nearest tenth of an inch, what should be the length $XY$ of a side so that it is tangent to the wheel?

44. Andrea and Carlos both mow lawns. Andrea charges $14.00 plus $6.25 per hour. Carlos charges $12.50 plus $6.50 per hour. If they both mow $h$ hours and Andrea earns more money than Carlos, what is the range of values of $h$? (Previous course)

A point is chosen randomly on $LR$.
Use the diagram to find the probability of each event. *(Lesson 9-6)*

45. The point is not on $MP$.
46. The point is on $LP$.
47. The point is on $MN$ or $PR$.
48. The point is on $QR$. 

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Circle Graphs

A circle graph compares data that are parts of a whole unit. When you make a circle graph, you find the measure of each central angle. A central angle is an angle whose vertex is the center of the circle.

See Skills Bank page S80

Example

Make a circle graph to represent the following data.

Step 1 Add all the amounts. \(110 + 40 + 300 + 150 = 600\)

Step 2 Write each part as a fraction of the whole.

- Fiction: \(\frac{110}{600}\)
- Nonfiction: \(\frac{40}{600}\)
- Children's: \(\frac{300}{600}\)
- Audio books: \(\frac{150}{600}\)

Step 3 Multiply each fraction by 360° to calculate the central angle measure.

- Fiction: \(\frac{110}{600}(360°) = 66°\)
- Nonfiction: \(\frac{40}{600}(360°) = 24°\)
- Children's: \(\frac{300}{600}(360°) = 180°\)
- Audio books: \(\frac{150}{600}(360°) = 90°\)

Step 4 Make a circle graph. Then color each section of the circle to match the data.

The section with a central angle of 66° is green, 24° is orange, 180° is purple, and 90° is yellow.

Try This

Choose the circle graph that best represents the data. Show each step.

1. Books in Linda's Library

   - Novels: 18
   - Reference: 10
   - Textbooks: 8

2. Vacation Expenses ($)

   - Travel: 450
   - Meals: 120
   - Lodging: 900
   - Other: 330

3. Puppy Expenses ($)

   - Food: 190
   - Health: 375
   - Training: 120
   - Other: 50

Data Analysis

Connecting Geometry to Data Analysis 755
11-2 Arcs and Chords

Objectives
Apply properties of arcs.
Apply properties of chords.

Vocabulary
- central angle
- arc
- minor arc
- major arc
- semicircle
- adjacent arcs
- congruent arcs

Who uses this?
Market analysts use circle graphs to compare sales of different products.

A central angle is an angle whose vertex is the center of a circle. An arc is an unbroken part of a circle consisting of two points called the endpoints and all the points on the circle between them.

### Arcs and Their Measure

<table>
<thead>
<tr>
<th>ARC</th>
<th>MEASURE</th>
<th>DIAGRAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>A minor arc</td>
<td>The measure of a minor arc is equal to the measure of its central angle.</td>
<td>![Diagram of minor arc]</td>
</tr>
<tr>
<td></td>
<td>( m\overparen{AC} = m\angle ABC = x^\circ )</td>
<td></td>
</tr>
<tr>
<td>A major arc</td>
<td>The measure of a major arc is equal to (360^\circ) minus the measure of its central angle.</td>
<td>![Diagram of major arc]</td>
</tr>
<tr>
<td></td>
<td>( m\overparen{AD} = 360^\circ - m\angle ABC )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( = 360^\circ - x^\circ )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>If the endpoints of an arc lie on a diameter, the arc is a semicircle.</td>
<td>![Diagram of semicircle]</td>
</tr>
<tr>
<td></td>
<td>The measure of a semicircle is equal to (180^\circ).</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( m\overparen{EF} = 180^\circ )</td>
<td></td>
</tr>
</tbody>
</table>

### Example 1

Data Application

The circle graph shows the types of music sold during one week at a music store.

Find \( m\overparen{BC} \).

\[
m\overparen{BC} = m\angle BMC
\]

\[
m\angle BMC = 0.13(360^\circ)
\]

\[
= 46.8^\circ
\]

Use the graph to find each of the following.

1a. \( m\angle FMC \)  
1b. \( m\overparen{AHB} \)  
1c. \( m\angle EMD \)
Adjacent arcs are arcs of the same circle that intersect at exactly one point. $RS$ and $ST$ are adjacent arcs.

**Postulate 11-2-1  Arc Addition Postulate**

The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.

$$m\widehat{ABC} = m\widehat{AB} + m\widehat{BC}$$

**Example 2** Using the Arc Addition Postulate

Find $m\widehat{CDE}$

- $m\widehat{CD} = 90^\circ$
- $m\angle CFD = 90^\circ$
- $m\angle DFE = 18^\circ$
- $m\widehat{DE} = 18^\circ$
- $m\widehat{CE} = m\widehat{CD} + m\widehat{DE}$

$$= 90^\circ + 18^\circ = 108^\circ$$

**Theorem 11-2-2**

<table>
<thead>
<tr>
<th>THEOREM</th>
<th>HYPOTHESIS</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>In a circle or congruent circles:</td>
<td>$\angle EAD \cong \angle BAC$</td>
<td>$DE \cong BC$</td>
</tr>
<tr>
<td>(1) Congruent central angles have congruent chords.</td>
<td>$ED \cong BC$</td>
<td>$DE \cong BC$</td>
</tr>
<tr>
<td>(2) Congruent chords have congruent arcs.</td>
<td>$ED \cong BC$</td>
<td>$\angle DAE \cong \angle BAC$</td>
</tr>
<tr>
<td>(3) Congruent arcs have congruent central angles.</td>
<td>$\angle DAE \cong \angle BAC$</td>
<td>$\angle DAE \cong \angle BAC$</td>
</tr>
</tbody>
</table>

You will prove parts 2 and 3 of Theorem 11-2-2 in Exercises 40 and 41.
The converses of the parts of Theorem 11-2-2 are also true. For example, with part 1, congruent chords have congruent central angles.

**Theorem 11-2-2 (Part 1)**

**Given:** \( \angle BAC \cong \angle DAE \)

**Prove:** \( BC \cong DE \)

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle BAC \cong \angle DAE )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \overline{AB} \cong \overline{AD}, \overline{AC} \cong \overline{AE} )</td>
<td>2. All radii of a ( \bigodot ) are ( \cong ).</td>
</tr>
<tr>
<td>3. ( \triangle BAC \cong \triangle DAE )</td>
<td>3. SAS Steps 2, 1</td>
</tr>
<tr>
<td>4. ( BC \cong DE )</td>
<td>4. CPCTC</td>
</tr>
</tbody>
</table>

**EXAMPLE 3**

**Applying Congruent Angles, Arcs, and Chords**

Find each measure.

**A** \( \overline{RS} \cong \overline{TU} \). Find \( m\overline{RS} \).

\[
\begin{align*}
\overline{RS} & \cong \overline{TU} & \cong \text{ chords have } \cong \text{ arcs.} \\
m\overline{RS} & = m\overline{TU} & \text{Def. of } \cong \text{ arcs} \\
3x & = 2x + 27 & \text{Substitute the given measures.} \\
x & = 27 & \text{Subtract } 2x \text{ from both sides.} \\
m\overline{RS} & = 3(27) & \text{Substitute } 27 \text{ for } x. \\
\text{ } & = 81^\circ & \text{Simplify.} \\
\end{align*}
\]

**B** \( \odot B \cong \odot E \), and \( \overparen{AC} \cong \overparen{DF} \). Find \( m\angle DEF \).

\[
\begin{align*}
\angle ABC & \cong \angle DEF & \cong \text{ arcs have } \cong \text{ central } \triangle. \\
m\angle ABC & = m\angle DEF & \text{Def. of } \cong \triangle \\
5y + 5 & = 7y - 43 & \text{Substitute the given measures.} \\
5 & = 2y - 43 & \text{Subtract } 5y \text{ from both sides.} \\
48 & = 2y & \text{Add } 43 \text{ to both sides.} \\
24 & = y & \text{Divide both sides by } 2. \\
m\angle DEF & = 7(24) - 43 & \text{Substitute } 24 \text{ for } y. \\
\text{ } & = 125^\circ & \text{Simplify.} \\
\end{align*}
\]

**Find each measure.**

**3a.** \( \overline{PT} \) bisects \( \angle RPS \). Find \( RT \).

**3b.** \( \odot A \cong \odot B \), and \( \overparen{CD} \cong \overparen{EF} \). Find \( m\overparen{CD} \).
**Theorems**

<table>
<thead>
<tr>
<th>THEOREM</th>
<th>HYPOTHESIS</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>11-2-3</strong></td>
<td>In a circle, if a radius (or diameter) is perpendicular to a chord, then it bisects the chord and its arc.</td>
<td>$\overline{CD}$ bisects $\overline{EF}$ and $\overline{EF}$.</td>
</tr>
<tr>
<td><strong>11-2-4</strong></td>
<td>In a circle, the perpendicular bisector of a chord is a radius (or diameter).</td>
<td>$\overline{JK}$ is a diameter of $\odot A$.</td>
</tr>
</tbody>
</table>

You will prove Theorems 11-2-3 and 11-2-4 in Exercises 42 and 43.

**Example 4**

**Using Radii and Chords**

Find $BD$.

**Step 1** Draw radius $\overline{AD}$.

$AD = 5$  
Radii of a $\odot$ are $\cong$.

**Step 2** Use the Pythagorean Theorem.

$CD^2 + AC^2 = AD^2$  
$CD^2 + 3^2 = 5^2$  
Substitute 3 for $AC$ and 5 for $AD$.  
$CD^2 = 16$  
Subtract $3^2$ from both sides.  
$CD = 4$  
Take the square root of both sides.

**Step 3** Find $BD$.

$BD = 2(4) = 8$  
$\overline{AE} \perp \overline{BD}$, so $\overline{AE}$ bisects $\overline{BD}$.

**Check It Out!**

4. Find $QR$ to the nearest tenth.

**Think and Discuss**

1. What is true about the measure of an arc whose central angle is obtuse?
2. Under what conditions are two arcs the same measure but not congruent?
3. **GET ORGANIZED** Copy and complete the graphic organizer. In each box, write a definition and draw a sketch.
GUIDED PRACTICE

Vocabulary: Apply the vocabulary from this lesson to answer each question.

1. An arc that joins the endpoints of a diameter is called a ____. (semicircle or major arc)

2. How do you recognize a central angle of a circle?

3. In \( \odot P \) \( \overparen{ABC} = 205^\circ \). Therefore \( \overparen{ABC} \) is a ____. (major arc or minor arc)

4. In a circle, an arc that is less than a semicircle is a ____. (major arc or minor arc)

Consumer Application: Use the following information for Exercises 5–10.

The circle graph shows how a typical household spends money on energy. Find each of the following.

5. \( m\angle PAQ \)
6. \( m\angle VAU \)
7. \( m\angle SAQ \)
8. \( m\overparen{UT} \)
9. \( m\overparen{RQ} \)
10. \( m\overparen{UPT} \)

Find each measure.

11. \( m\overparen{DF} \)
12. \( m\overparen{DEB} \)
13. \( m\overparen{L} \)
14. \( m\overparen{HLK} \)
15. \( \angle QPR \cong \angle RPS \). Find \( QR \).
16. \( \odot A \cong \odot B \), and \( \overparen{CD} \cong \overparen{EF} \). Find \( m\angle EBF \).

Multi-Step: Find each length to the nearest tenth.

17. \( RS \)
18. \( EF \)
**PRACTICE AND PROBLEM SOLVING**

**Sports** Use the following information for Exercises 19–24.

The key shows the number of medals won by U.S. athletes at the 2004 Olympics in Athens. Find each of the following to the nearest tenth.

19. \( m \angle ADB \)
20. \( m \angle ADC \)
21. \( \overarc{AB} \)
22. \( \overarc{BC} \)
23. \( \overarc{ACB} \)
24. \( \overarc{CAB} \)

Find each measure.

25. \( \overarc{MP} \)
26. \( \overarc{QNL} \)

27. \( m \overarc{WT} \)
28. \( m \overarc{WTV} \)

29. \( \odot A \cong \odot B \), and \( \overarc{CD} \cong \overarc{EF} \).

Find \( m \angle CAD \).

30. \( \overline{JK} \parallel \overline{LM} \). Find \( m \angle J \).

**Multi-Step** Find each length to the nearest tenth.

31. \( \overline{CD} \)
32. \( \overline{RS} \)

Determine whether each statement is true or false. If false, explain why.

33. The central angle of a minor arc is an acute angle.
34. Any two points on a circle determine a minor arc and a major arc.
35. In a circle, the perpendicular bisector of a chord must pass through the center of the circle.

36. **Data Collection** Use a graphing calculator, a pH probe, and a data-collection device to collect information about the pH levels of ten different liquids. Then create a circle graph with the following sectors: strong basic \((9 < \text{pH} < 14)\), weak basic \((7 < \text{pH} < 9)\), neutral \((\text{pH} = 7)\), weak acidic \((5 < \text{pH} < 7)\), and strong acidic \((0 < \text{pH} < 5)\).

37. In \( \odot E \), the measures of \( \angle AEB \), \( \angle BEC \), and \( \angle CED \) are in the ratio 3:4:5. Find \( m \overarc{AB} \), \( m \overarc{BC} \), and \( m \overarc{CD} \).
**Algebra** Find the indicated measure.

38. \( m\widehat{IL} \) 

\[
\begin{align*}
(4x - 2)^\circ & = (7x - 18)^\circ \\
(6x + 6)^\circ & 
\end{align*}
\]

39. \( m\angle SPT \)

\[
\begin{align*}
10^\circ & = 6z^\circ 
\end{align*}
\]

40. Prove \( \cong \) chords have \( \cong \) arcs.

**Given:** \( \odot A, \widehat{BC} \cong \widehat{DE} \)

**Prove:** \( \widehat{BC} \cong \widehat{DE} \)

41. Prove \( \cong \) arcs have \( \cong \) central \( \triangle \).

**Given:** \( \odot A, \widehat{BC} \cong \widehat{DE} \)

**Prove:** \( \angle BAC \cong \angle DAE \)

42. Prove Theorem 11-2-3.

**Given:** \( \odot C, \overline{CD} \perp \overline{EF} \)

**Prove:** \( \overline{CD} \) bisects \( \overline{EF} \) and \( \overline{EF} \).

(Hint: Draw \( \overline{CE} \) and \( \overline{CF} \) and use the HL Theorem.)

43. Prove Theorem 11-2-4.

**Given:** \( \odot A, \overline{JK} \perp \) bisector of \( \overline{GH} \)

**Prove:** \( \overline{JK} \) is a diameter

(Hint: Use the Converse of the \( \perp \) Bisector Theorem.)

44. **Critical Thinking** Roberto folds a circular piece of paper as shown. When he unfolds the paper, how many different-sized central angles will be formed?

![One fold Two folds Three folds](image)

45. ///ERROR ANALYSIS/// Below are two solutions to find the value of \( x \). Which solution is incorrect? Explain the error.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overline{AD} ) is a diam., so ( m\widehat{ACD} = 180^\circ ). ( m\widehat{AB} + m\widehat{BC} + m\widehat{CD} = 180^\circ ). ( 5x + 90 + 15x = 180 ). 20x = 90, ( x = 4.5 ).</td>
<td>Because they are vert. ( \triangle ), ( \angle AGF \cong \angle CGD ). Thus ( m\widehat{AF} = m\widehat{CD} ). ( 16x - 5 = 15x ). ( x = 5 ).</td>
</tr>
</tbody>
</table>

46. **Write About It** According to a school survey, 40% of the students take a bus to school, 35% are driven to school, 15% ride a bike, and the remainder walk. Explain how to use central angles to create a circle graph from this data.

47. This problem will prepare you for the Multi-Step Test Prep on page 770.

Chantal’s bike has wheels with a 27 in. diameter.

a. What are \( AC \) and \( AD \) if \( DB \) is 7 in.?

b. What is \( CD \) to the nearest tenth of an inch?

c. What is \( CE \), the length of the top of the bike stand?
48. Which of these arcs of \( \odot Q \) has the greatest measure?
- A \( \overset{\frown}{WT} \)
- B \( \overset{\frown}{UW} \)
- C \( \overset{\frown}{VR} \)
- D \( \overset{\frown}{TV} \)

49. In \( \odot A \), \( CD = 10 \). Which of these is closest to the length of \( \overset{\frown}{AE} \)?
- F 3.3 cm
- G 4 cm
- H 5 cm
- I 7.8 cm

50. **Gridded Response** \( \odot P \) has center \( (2, 1) \) and radius 3. What is the measure, in degrees, of the minor arc with endpoints \( A(-1, 1) \) and \( B(2, -2) \)?

### Challenge and Extend

51. In the figure, \( AB \perp CD \). Find \( m\overset{\frown}{BD} \) to the nearest tenth of a degree.

52. Two points on a circle determine two distinct arcs. How many arcs are determined by \( n \) points on a circle? (Hint: Make a table and look for a pattern.)

53. An angle measure other than degrees is **radian** measure. \( 360^\circ \) converts to \( 2\pi \) radians, or \( 180^\circ \) converts to \( \pi \) radians.
   a. Convert the following radian angle measures to degrees: \( \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4} \).
   b. Convert the following angle measures to radians: \( 135^\circ, 270^\circ \).

### Spiral Review

Simplify each expression. *(Previous course)*

54. \( (3x)^3(2y^2)(3^{-2}y^2) \)

55. \( a^4b^3(-2a)^{-4} \)

56. \( (-2t^3s^2)(3ts^2)^2 \)

Find the next term in each pattern. *(Lesson 2-1)*

57. 1, 3, 7, 13, 21, ...

58. C, E, G, I, K, ...

59. 1, 6, 15, ...

In the figure, \( \overline{QP} \) and \( \overline{QM} \) are tangent to \( \odot N \). Find each measure. *(Lesson 11-1)*

60. \( m\angle NMQ \)

61. \( MQ \)

### Construction

**Circle Through Three Noncollinear Points**

1. Draw three noncollinear points.

2. Construct \( m \) and \( n \), the \( \perp \) bisectors of \( \overline{PQ} \) and \( \overline{QR} \). Label the intersection \( O \).

3. Center the compass at \( O \). Draw a circle through \( P \).

1. Explain why \( \odot O \) with radius \( OP \) also contains \( Q \) and \( R \).
Who uses this?
Farmers use irrigation radii to calculate areas of sectors. (See Example 2.)

The area of a sector is a fraction of the circle containing the sector. To find the area of a sector whose central angle measures \( m^\circ \), multiply the area of the circle by \( \frac{m^\circ}{360^\circ} \).

**Example 1**

Finding the Area of a Sector

Find the area of each sector. Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.

**A** sector \( MPN \)

\[
A = \pi r^2 \left( \frac{m^\circ}{360^\circ} \right)
\]

\[
= \pi (3)^2 \left( \frac{80^\circ}{360^\circ} \right)
\]

\[
= 2\pi \text{ in}^2 \approx 6.28 \text{ in}^2
\]

**B** sector \( EFG \)

\[
A = \pi r^2 \left( \frac{m^\circ}{360^\circ} \right)
\]

\[
= \pi (6)^2 \left( \frac{120^\circ}{360^\circ} \right)
\]

\[
= 12\pi \approx 37.70 \text{ cm}^2
\]

Find the area of each sector. Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.

1a. sector \( ACB \)

1b. sector \( JKL \)
Agriculture Application

A circular plot with a 720 ft diameter is watered by a spray irrigation system. To the nearest square foot, what is the area that is watered as the sprinkler rotates through an angle of 50°?

\[ A = \pi r^2 \left( \frac{m^\circ}{360^\circ} \right) \]

\[ = \pi (360)^2 \left( \frac{50^\circ}{360^\circ} \right) \]

\[ d = 720 \text{ ft, } r = 360 \text{ ft.} \]

\[ \approx 56,549 \text{ ft}^2 \]

Simplify.

To the nearest square foot, what is the area watered in Example 2 as the sprinkler rotates through a semicircle?

A segment of a circle is a region bounded by an arc and its chord. The shaded region in the figure is a segment.

Area of a Segment

\[ \text{area of segment} = \text{area of sector} - \text{area of triangle} \]

Example 3

Finding the Area of a Segment

Find the area of segment \(ACB\) to the nearest hundredth.

Step 1 Find the area of sector \(ACB\).

\[ A = \pi r^2 \left( \frac{m^\circ}{360^\circ} \right) \]

Use formula for area of a sector.

\[ = \pi (12)^2 \left( \frac{60^\circ}{360^\circ} \right) \]

Substitute 12 for \(r\) and 60 for \(m\).

\[ = 24\pi \text{ in}^2 \]

Step 2 Find the area of \(\triangle ACB\).

Draw altitude \(\overline{AD}\).

\[ A = \frac{1}{2}bh = \frac{1}{2}(12)(6\sqrt{3}) \]

\[ = 36\sqrt{3} \text{ in}^2 \]

\[ CD = 6 \text{ in., and } h = 6\sqrt{3} \text{ in.} \]

Simplify.

Step 3 area of segment = area of sector \(ACB\) – area of \(\triangle ACB\)

\[ = 24\pi - 36\sqrt{3} \]

\[ \approx 13.04 \text{ in}^2 \]

Check it out!

Find the area of segment \(RST\) to the nearest hundredth.
In the same way that the area of a sector is a fraction of the area of the circle, the length of an arc is a fraction of the circumference of the circle.

### Arc Length

<table>
<thead>
<tr>
<th>TERM</th>
<th>DIAGRAM</th>
<th>LENGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arc length</td>
<td><img src="arc-length.png" alt="Diagram" /></td>
<td>$L = 2\pi r \left( \frac{m^\circ}{360^\circ} \right)$</td>
</tr>
</tbody>
</table>

#### Example 4: Finding Arc Length

Find each arc length. Give your answer in terms of $\pi$ and rounded to the nearest hundredth.

**A**

Find arc $\overparen{CD}$.

$$L = 2\pi r \left( \frac{m^\circ}{360^\circ} \right)$$

Use formula for arc length.

$= 2\pi(10) \left( \frac{90^\circ}{360^\circ} \right)$

Substitute $10$ for $r$ and $90$ for $m$.

$= 5\pi \approx 15.71$ ft

Simplify.

**B**

An arc with measure $35^\circ$ in a circle with radius $3$ in.

$$L = 2\pi r \left( \frac{m^\circ}{360^\circ} \right)$$

Use formula for arc length.

$= 2\pi(3) \left( \frac{35^\circ}{360^\circ} \right)$

Substitute $3$ for $r$ and $35$ for $m$.

$= \frac{7}{12}$ in. $\approx 1.83$ in.

Simplify.

Find each arc length. Give your answer in terms of $\pi$ and rounded to the nearest hundredth.

4a. $\overparen{GH}$

4b. An arc with measure $135^\circ$ in a circle with radius $4$ cm

#### Check It Out!

Find each arc length. Give your answer in terms of $\pi$ and rounded to the nearest hundredth.

**4a.** $\overparen{GH}$

**4b.** An arc with measure $135^\circ$ in a circle with radius $4$ cm

### Think and Discuss

1. What is the difference between arc measure and arc length?
2. A slice of pizza is a sector of a circle. Explain what measurements you would need to make in order to calculate the area of the slice.
3. **Get Organized** Copy and complete the graphic organizer.

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area of a Sector</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Area of a Segment</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Arc Length</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
11-3 Sector Area and Arc Length

GUIDED PRACTICE

1. **Vocabulary** In a circle, the region bounded by a chord and an arc is called a __?___. (sector or segment)

Find the area of each sector. Give your answer in terms of $\pi$ and rounded to the nearest hundredth.

2. sector $PQR$

3. sector $JKL$

4. sector $ABC$

5. **Navigation** The beam from a lighthouse is visible for a distance of $3$ mi. To the nearest square mile, what is the area covered by the beam as it sweeps in an arc of $150^\circ$?

Multi-Step Find the area of each segment to the nearest hundredth.

6. 

7. 

8. 

9. Find each arc length. Give your answer in terms of $\pi$ and rounded to the nearest hundredth.

10. 

11. an arc with measure $20^\circ$ in a circle with radius $6$ in.

PRACTICE AND PROBLEM SOLVING

Find the area of each sector. Give your answer in terms of $\pi$ and rounded to the nearest hundredth.

12. sector $DEF$

13. sector $GHJ$

14. sector $RST$

15. **Architecture** A lunette is a semicircular window that is sometimes placed above a doorway or above a rectangular window. To the nearest square inch, what is the area of the lunette?
Find the area of each segment to the nearest hundredth.

16. \( \triangle ABC \)
   \[ \begin{array}{c}
   A
   \end{array} \]

17. \( \triangle LKM \)
   \[ \begin{array}{c}
   L
   \end{array} \]

18. \( \triangle RST \)
   \[ \begin{array}{c}
   R
   \end{array} \]

Find each arc length. Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.

19. \( \overline{UV} \)
   \[ \begin{array}{c}
   U
   \end{array} \]

20. \( \overline{AB} \)
   \[ \begin{array}{c}
   A
   \end{array} \]

21. an arc with measure 9° in a circle with diameter 4 ft

22. **Math History** Greek mathematicians studied the salinon, a figure bounded by four semicircles. What is the perimeter of this salinon to the nearest tenth of an inch?

Tell whether each statement is sometimes, always, or never true.

23. The length of an arc of a circle is greater than the circumference of the circle.

24. Two arcs with the same measure have the same arc length.

25. In a circle, two arcs with the same length have the same measure.

Find the radius of each circle.

26. area of sector \( ABC = 9\pi \)
   \[ \begin{array}{c}
   A
   \end{array} \]

27. arc length of \( EF = 8\pi \)
   \[ \begin{array}{c}
   E
   \end{array} \]

28. **Estimation** The fraction \( \frac{22}{7} \) is an approximation for \( \pi \).
   a. Use this value to estimate the arc length of \( \overline{XY} \).
   b. Use the \( \pi \) key on your calculator to find the length of \( \overline{XY} \) to 8 decimal places.
   c. Was your estimate in part a an overestimate or an underestimate?

29. This problem will prepare you for the Multi-Step Test Prep on page 770.
   The pedals of a penny-farthing bicycle are directly connected to the front wheel.
   a. Suppose a penny-farthing bicycle has a front wheel with a diameter of 5 ft. To the nearest tenth of a foot, how far does the bike move when you turn the pedals through an angle of 90°?
   b. Through what angle should you turn the pedals in order to move forward by a distance of 4.5 ft? Round to the nearest degree.
30. **Critical Thinking** What is the length of the radius that makes the area of \( \odot A = 24 \text{ in}^2 \) and the area of sector \( BAC = 3 \text{ in}^2 \)? Explain.

31. **Write About It** Given the length of an arc of a circle and the measure of the arc, explain how to find the radius of the circle.

32. **What is the area of sector \( AOB \)?**

   - \( \text{A} 4\pi \)
   - \( \text{B} 16\pi \)
   - \( \text{C} 32\pi \)
   - \( \text{D} 64\pi \)

33. **What is the length of \( \widehat{AB} \)?**

   - \( \text{F} 2\pi \)
   - \( \text{G} 4\pi \)
   - \( \text{H} 8\pi \)
   - \( \text{I} 16\pi \)

34. **Gridded Response** To the nearest hundredth, what is the area of the sector determined by an arc with measure 35° in a circle with radius 12?

### CHALLENGE AND EXTEND

35. In the diagram, the larger of the two concentric circles has radius 5, and the smaller circle has radius 2. What is the area of the shaded region in terms of \( \pi \)?

36. A wedge of cheese is a sector of a cylinder.
   a. To the nearest tenth, what is the volume of the wedge with the dimensions shown?
   b. What is the surface area of the wedge of cheese to the nearest tenth?

37. **Probability** The central angles of a target measure 45°. The inner circle has a radius of 1 ft, and the outer circle has a radius of 2 ft. Assuming that all arrows hit the target at random, find the following probabilities.
   a. hitting a red region
   b. hitting a blue region
   c. hitting a red or blue region

### SPIRAL REVIEW

Determine whether each line is parallel to \( y = 4x - 5 \), perpendicular to \( y = 4x - 5 \), or neither. *(Previous course)*

38. \( 8x - 2y = 6 \)

39. line passing through the points \( \left( \frac{1}{2}, 0 \right) \) and \( \left( 1 \frac{1}{2}, 2 \right) \)

40. line with \( x \)-intercept 4 and \( y \)-intercept 1

Find each measurement. Give your answer in terms of \( \pi \). *(Lesson 10-8)*

41. volume of a sphere with radius 3 cm

42. circumference of a great circle of a sphere whose surface area is \( 4\pi \text{ cm}^2 \)

Find the indicated measure. *(Lesson 11-2)*

43. \( m\angle K LJ \)

44. \( m\overarc{JK} \)

45. \( m\angle FH \)
Lines and Arcs in Circles

As the Wheels Turn  The bicycle was invented in the 1790s. The first models didn’t even have pedals—riders moved forward by pushing their feet along the ground! Today the bicycle is a high-tech machine that can include hydraulic brakes and electronic gear changers.

1. A road race bicycle wheel is 28 inches in diameter. A manufacturer makes metal bicycle stands that are 10 in. tall. How long should a stand be to the nearest tenth in order to support a 28 in. wheel? (Hint: Consider the triangle formed by the radii and the top of the stand.)

2. The chain of a bicycle loops around a large gear connected to the bike’s pedals and a small gear attached to the rear wheel. In the diagram, the distance $AB$ between the centers of the gears the nearest tenth is 15 in. Find $CD$, the length of the chain between the two gears to the nearest tenth. (Hint: Draw a segment from $B$ to $AD$ that is parallel to $CD$.)

3. By pedaling, you turn the large gear through an angle of 60°. How far does the chain move around the circumference of the gear to the nearest tenth?

4. As the chain moves, it turns the small gear. If you use the distance you calculated in Problem 3, through what angle does the small gear turn to the nearest degree?
Quiz for Lessons 11-1 Through 11-3

11-1 Lines That Intersect Circles
Identify each line or segment that intersects each circle.

1. 

2. 

3. The tallest building in Africa is the Carlton Centre in Johannesburg, South Africa. What is the distance from the top of this 732 ft building to the horizon to the nearest mile? (Hint: 5280 ft = 1 mi; radius of Earth = 4000 mi)

11-2 Arcs and Chords
Find each measure.

4. \(\widehat{BC}\)

5. \(\overline{BED}\)

6. \(\widehat{SR}\)

7. \(\overline{SQU}\)

Find each length to the nearest tenth.

8. \(\overline{JK}\)

9. \(\overline{XY}\)

11-3 Sector Area and Arc Length

10. As part of an art project, Peter buys a circular piece of fabric and then cuts out the sector shown. What is the area of the sector to the nearest square centimeter?

Find each arc length. Give your answer in terms of \(\pi\) and rounded to the nearest hundredth.

11. \(\overline{AB}\)

12. \(\overline{EF}\)

13. an arc with measure 44° in a circle with diameter 10 in.

14. a semicircle in a circle with diameter 92 m
Objectives
Find the measure of an inscribed angle.
Use inscribed angles and their properties to solve problems.

Vocabulary
inscribed angle
intercepted arc
subtend

Why learn this?
You can use inscribed angles to find measures of angles in string art. (See Example 2.)

String art often begins with pins or nails that are placed around the circumference of a circle. A long piece of string is then wound from one nail to another. The resulting pattern may include hundreds of inscribed angles.

An inscribed angle is an angle whose vertex is on a circle and whose sides contain chords of the circle. An intercepted arc consists of endpoints that lie on the sides of an inscribed angle and all the points of the circle between them. A chord or arc subtends an angle if its endpoints lie on the sides of the angle.

\[ \angle DEF \text{ is an inscribed angle.} \]
\[ \widehat{DF} \text{ is the intercepted arc.} \]
\[ \widehat{DF} \text{ subtends } \angle DEF. \]

The measure of an inscribed angle is half the measure of its intercepted arc.

\[ m\angle ABC = \frac{1}{2} m\widehat{AC} \]

You will prove Cases 2 and 3 of Theorem 11-4-1 in Exercises 30 and 31.

Inscribed Angle Theorem

PROOF

Given: \( \angle ABC \) is inscribed in \( \odot X \).
Prove: \( m\angle ABC = \frac{1}{2} m\widehat{AC} \)

Proof Case 1:
\( \angle ABC \) is inscribed in \( \odot X \) with \( X \) on \( BC \). Draw \( \overline{XA} \). \( m\widehat{AC} = m\angle XAC \).
By the Exterior Angle Theorem \( m\angle XAC = m\angle ABX + m\angle BAX \).
Since \( \overline{XA} \) and \( \overline{XB} \) are radii of the circle, \( \overline{XA} \cong \overline{XB} \). Then by definition \( \triangle AXB \) is isosceles. Thus \( m\angle ABX = m\angle BAX \).

By the Substitution Property, \( m\widehat{AC} = 2m\angle ABX \) or \( 2m\angle ABC \).
Thus \( \frac{1}{2} m\widehat{AC} = m\angle ABC. \)
EXAMPLE 1  Finding Measures of Arcs and Inscribed Angles

Find each measure.

A. \( m\angle RST \)
   \[
   m\angle RST = \frac{1}{2} m\widehat{RT} \quad \text{Inscribed \( \angle \) Thm.}
   \]
   \[
   = \frac{1}{2} (120^\circ) = 60^\circ \quad \text{Substitute 120 for \( m\widehat{RT} \).}
   \]

B. \( m\widehat{SU} \)
   \[
   m\angle SRU = \frac{1}{2} m\widehat{SU} \quad \text{Inscribed \( \angle \) Thm.}
   \]
   \[
   40^\circ = \frac{1}{2} m\widehat{SU} \quad \text{Substitute 40 for \( m\angle SRU \).}
   \]
   \[
   m\widehat{SU} = 80^\circ \quad \text{Mult. both sides by 2.}
   \]

Find each measure.

1a. \( m\widehat{ADC} \)
1b. \( m\angle DAE \)

COROLLARY 11-4-2

If inscribed angles of a circle intercept the same arc or are subtended by the same chord or arc, then the angles are congruent.

\[
\angle ACB \cong \angle ADB \cong \angle AEB
\]

\[
\angle ACB, \angle ADB, \text{ and } \angle AEB \text{ intercept } AB.
\]

You will prove Corollary 11-4-2 in Exercise 32.

EXAMPLE 2  Hobby Application

Find \( m\angle DEC \), if \( m\widehat{AD} = 86^\circ \).

\[
\angle BAC \cong \angle BDC \quad \triangle \text{BAC and } \angle BDC \text{ intercept } BC.
\]
\[
m\angle BAC = m\angle BDC \quad \text{Def. of } \cong
\]
\[
m\angle BDC = 60^\circ \quad \text{Substitute 60 for } \angle BDC.
\]
\[
m\angle ACD = \frac{1}{2} m\widehat{AD} \quad \text{Inscribed } \angle \text{ Thm.}
\]
\[
= \frac{1}{2} (86^\circ) \quad \text{Substitute 86 for } m\widehat{AD}.
\]
\[
= 43^\circ \quad \text{Simplify.}
\]
\[
m\angle DEC + 60 + 43 = 180 \quad \triangle \text{Sum Theorem}
\]
\[
m\angle DEC = 77^\circ \quad \text{Simplify.}
\]

2. Find \( m\angle ABD \) and \( m\widehat{BC} \) in the string art.
An inscribed angle subtends a semicircle if and only if the angle is a right angle.

You will prove Theorem 11-4-3 in Exercise 43.

**Example 3**

**Finding Angle Measures in Inscribed Triangles**

Find each value.

**A**

$x$

∠RQT is a right angle

∠RQT is inscribed in a semicircle.

$m\angle RQT = 90°$

Def. of rt. ∠

$4x + 6 = 90$

Substitute $4x + 6$ for $m\angle RQT$.

$4x = 84$

Subtract 6 from both sides.

$x = 21$

Divide both sides by 4.

**B**

$m\angle ADC$

$m\angle ABC = m\angle ADC$

∠ABC and ∠ADC both intercept ⃗AC.

$10y - 28 = 7y - 1$

Substitute the given values.

$3y - 28 = -1$

Subtract 7y from both sides.

$3y = 27$

Add 28 to both sides.

$y = 9$

Divide both sides by 3.

$m\angle ADC = 7(9) - 1 = 62°$

Substitute 9 for y.

Find each value.

3a. $z$

3b. $m\angle EDF$

**Construction**

Center of a Circle

1. Draw a circle and chord $\overline{AB}$.
2. Construct a line perpendicular to $\overline{AB}$ at $B$. Where the line and the circle intersect, label the point $C$.
3. Draw chord $\overline{AC}$.
4. Repeat steps to draw chords $\overline{DE}$ and $\overline{DF}$. The intersection of $\overline{AC}$ and $\overline{DF}$ is the center of the circle.
### Theorem 11-4-4

**Theorem**

If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

**Hypothesis**

\[ \angle A \text{ and } \angle C \text{ are supplementary.} \]
\[ \angle B \text{ and } \angle D \text{ are supplementary.} \]

**Conclusion**

\[ ABCD \text{ is inscribed in } \odot E. \]

You will prove Theorem 11-4-4 in Exercise 44.

**Example 4**

**Finding Angle Measures in Inscribed Quadrilaterals**

Find the angle measures of \( PQRS \).

**Step 1** Find the value of \( y \).

\[
\begin{align*}
\text{PQRS is inscribed in a } \odot. \\
6y + 1 + 10y + 19 &= 180 \\
16y + 20 &= 180 \\
16y &= 160 \\
y &= 10
\end{align*}
\]

**Step 2** Find the measure of each angle.

\[
\begin{align*}
m\angle P &= 6(10) + 1 = 61^\circ \\
m\angle R &= 10(10) + 19 = 119^\circ \\
m\angle Q &= 10^\circ + 48 = 148^\circ \\
m\angle Q + m\angle S &= 180^\circ \\
m\angle Q \text{ and } \angle S &\text{ are supp.} \\
148^\circ + m\angle S &= 180^\circ \\
m\angle S &= 32^\circ
\end{align*}
\]

4. Find the angle measures of \( JKLM \).

**Think and Discuss**

1. Can \( \square ABCD \) be inscribed in a circle? Why or why not?
2. An inscribed angle intercepts an arc that is \( \frac{1}{4} \) of the circle. Explain how to find the measure of the inscribed angle.
3. **Get Organized** Copy and complete the graphic organizer. In each box write a definition, properties, an example, and a nonexample.
1. **Vocabulary** A, B, and C lie on ⊙P. ∠ABC is an example of an __?__ angle. (intercepted or inscribed)

Find each measure.

2. \( m\angle DEF \)

3. \( m\overarc{EG} \)

4. \( m\overarc{KL} \)

5. \( m\angle LKM \)

6. **Crafts** A circular loom can be used for knitting. What is the \( m\angle QTR \) in the knitting loom?

Find each value.

7. \( x \)

8. \( y \)

9. \( m\angle XYZ \)

10. **Multi-Step** Find the angle measures of each quadrilateral.

    - \( PQRS \)
    - \( ABCD \)

Find each measure.

12. \( m\overarc{ML} \)

13. \( m\angle KMN \)

14. \( m\overarc{EGH} \)

15. \( m\angle GFH \)

16. **Crafts** An artist created a stained glass window. If \( m\angle BEC = 40^\circ \) and \( m\angle AB = 44^\circ \), what is \( m\angle ADC \)?
**Algebra** Find each value.

17. \( y \)

\[
(3y^2 - 18)^\circ
\]

18. \( z \)

\[
(6z - 4)^\circ
\]

19. \( \overline{AB} \)

\[
(2x^2)^\circ
\]

20. \( \angle MPN \)

\[
(3x - 10)^\circ
\]

\[
\frac{11x}{3}\circ
\]

**Multi-Step** Find the angle measures of each quadrilateral.

21. \( BCDE \)

\[
(z - 59)^\circ
\]

\[
\left(\frac{x}{2}\right)^\circ
\]

\[
\left(\frac{x}{2} + 30\right)^\circ
\]

22. \( TUVW \)

\[
(15y - 4)^\circ
\]

\[
(14 + 4x)^\circ
\]

\[
(12y - 5)^\circ
\]

Tell whether each statement is sometimes, always, or never true.

23. Two inscribed angles that intercept the same arc of a circle are congruent.

24. When a right triangle is inscribed in a circle, one of the legs of the triangle is a diameter of the circle.

25. A trapezoid can be inscribed in a circle.

**Multi-Step** Find each angle measure.

26. \( \angle ABC \) if \( \angle ADC = 112^\circ \)

27. \( \angle PQR \) if \( \overparen{PQR} = 130^\circ \)

28. Prove that the measure of a central angle subtended by a chord is twice the measure of the inscribed angle subtended by the chord.

**Given:** In \( \odot H \), \( \overparen{JK} \) subtends \( \angle JHK \) and \( \angle JLK \).

**Prove:** \( \angle JHK = 2\angle JLK \)

29. This problem will prepare you for the Multi-Step Test Prep on page 806.

A Native American sand painting could be used to indicate the direction of sunrise on the winter and summer solstices. You can make this design by placing six equally spaced points around the circumference of a circle and connecting them as shown.

a. Find \( \angle BAC \).

b. Find \( \angle CDE \).

c. What type of triangle is \( \triangle FBC \)? Why?
30. **Given:** \(\angle ABC\) is inscribed in \(\odot X\) with \(X\) in the interior of \(\angle ABC\).
   **Prove:** \(m\angle ABC = \frac{1}{2}m\hat{AC}\)
   
   *(Hint: Draw \(B\bar{X}\) and use Case 1 of the Inscribed Angle Theorem.)*

31. **Given:** \(\angle ABC\) is inscribed in \(\odot X\) with \(X\) in the exterior of \(\angle ABC\).
   **Prove:** \(m\angle ABC = \frac{1}{2}m\hat{AC}\)

32. **Prove Corollary 11-4-2.**

33. **Multi-Step** In the diagram, \(m\hat{KL} = 198^\circ\), and \(m\hat{KL} = 216^\circ\). Find the measures of the angles of quadrilateral \(JKLM\).

34. **Critical Thinking** A rectangle \(PQRS\) is inscribed in a circle. What can you conclude about \(\overline{PR}\)? Explain.

35. **History** The diagram shows the Winchester Round Table with inscribed \(\triangle ABC\). The table may have been made at the request of King Edward III, who created the Order of Garter as a return to the Round Table and an order of chivalry.
   
   **a.** Explain why \(\overline{BC}\) must be a diameter of the circle.
   
   **b.** Find \(m\hat{AC}\).

36. To inscribe an equilateral triangle in a circle, draw a diameter \(\overline{BC}\). Open the compass to the radius of the circle. Place the point of the compass at \(C\) and make arcs on the circle at \(D\) and \(E\), as shown. Draw \(\overline{BD}, \overline{BE},\) and \(\overline{DE}\). Explain why \(\triangle BDE\) is an equilateral triangle.

37. **Write About It** A student claimed that if a parallelogram contains a 30° angle, it cannot be inscribed in a circle. Do you agree or disagree? Explain.

38. **Construction** Circumscribe a circle about a triangle. *(Hint: Follow the steps for the construction of a circle through three given noncollinear points.)*

39. What is \(m\angle BAC\)?
   
   - **A** 38°
   - **B** 43°
   - **C** 66°
   - **D** 81°

40. Equilateral \(\triangle XCZ\) is inscribed in a circle. If \(\overline{CV}\) bisects \(\angle C\), what is \(m\hat{XY}\)?
   
   - **F** 15°
   - **G** 30°
   - **H** 60°
   - **J** 120°

41. Quadrilateral \(ABCD\) is inscribed in a circle. The ratio of \(m\angle A\) to \(m\angle C\) is 4:5. What is \(m\angle A\)?
   
   - **A** 20°
   - **B** 40°
   - **C** 80°
   - **D** 100°

42. Which of these angles has the greatest measure?
   
   - **F** \(\angle STR\)
   - **G** \(\angle QPR\)
   - **H** \(\angle QSR\)
   - **I** \(\angle POS\)
   - **J** \(84°\)
   - **K** \(112°\)
**CHALLENGE AND EXTEND**

43. Prove that an inscribed angle subtends a semicircle if and only if the angle is a right angle. (*Hint: There are two parts.*)

44. Prove that if a quadrilateral is inscribed in a circle, then its opposite angles are supplementary. (*Hint: There are two parts.*)

45. Find \( \overline{PQ} \) to the nearest degree.

46. Find \( \angle BAC \).

47. **Construction** To circumscribe an equilateral triangle about a circle, construct \( AB \) parallel to the horizontal diameter of the circle and tangent to the circle. Then use a 30°-60°-90° triangle to draw \( AC \) and \( BC \) so that they form 60° angles with \( AB \) and are tangent to the circle.

**SPIRAL REVIEW**

48. Tickets for a play cost $15.00 for section C, $22.50 for section B, and $30.00 for section A. Amy spent a total of $255.00 for 12 tickets. If she spent the same amount on section C tickets as section A tickets, how many tickets for section B did she purchase? *(Previous course)*

Write a ratio expressing the slope of the line through each pair of points. *(Lesson 7-1)*

49. \( \left( \frac{4}{2}, -6 \right) \) and \( \left( \frac{8}{1}, \frac{1}{2} \right) \)

50. \( (-9, -8) \) and \( (0, -2) \)

51. \( (3, -14) \) and \( (11, 6) \)

Find each of the following. *(Lesson 11-2)*

52. \( m\overline{ST} \)

53. Area of \( \triangle ABD \)

**Construction** Tangent to a Circle From an Exterior Point

1. Can you draw \( \overline{CR} \perp \overline{RP} \)? Explain.
Explore Angle Relationships in Circles

In Lesson 11-4, you learned that the measure of an angle inscribed in a circle is half the measure of its intercepted arc. Now you will explore other angles formed by pairs of lines that intersect circles.

**Activity 1**

1. Create a circle with center A. Label the point on the circle as B. Create a radius segment from A to a new point C on the circle.

2. Construct a line through C perpendicular to radius AC. Create a new point D on this line, which is tangent to circle A at C. Hide radius AC.

3. Create a new point E on the circle and then construct secant CE.

4. Measure \( \angle DCE \) and measure \( \overarc{CBE} \).
   (Hint: To measure an arc in degrees, select the three points and the circle and then choose Arc Angle from the Measure menu.)

5. Drag E around the circle and examine the changes in the measures. Fill in the angle and arc measures in a chart like the one below. Try to create acute, right, and obtuse angles. Can you make a conjecture about the relationship between the angle measure and the arc measure?

<table>
<thead>
<tr>
<th>( m \angle DCE )</th>
<th>( m \overarc{CBE} )</th>
<th>Angle Type</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Activity 2**

1. Construct a new circle with two secants \( \overline{CD} \) and \( \overline{EF} \) that intersect inside the circle at G.

2. Create two new points H and I that are on the circle as shown. These will be used to measure the arcs. Hide B if desired. (It controls the circle’s size.)

3. Measure \( \angle DGF \) formed by the secant lines and measure \( \overarc{CHI} \) and \( \overarc{DIF} \).

4. Drag F around the circle and examine the changes in measures. Be sure to keep H between C and E and I between D and F for accurate arc measurement. Move them if needed.
Try This

1. How does the relationship you observed in Activity 1 compare to the relationship between an inscribed angle and its intercepted arc?

2. Why do you think the radius $\overline{AC}$ is needed in Activity 1 for the construction of the tangent line? What theorem explains this?

3. In Activity 3, try dragging points so that the secants become tangents. What conclusion can you make about the angle and arc measures?

4. Examine the conjectures and theorems about the relationships between angles and arcs in a circle. What is true of an angle with a vertex on the circle? What is true of an angle with a vertex inside the circle? What is true of an angle with a vertex outside the circle? Summarize your findings.

5. Does using geometry software to compare angle and arc measures constitute a formal proof of the relationship observed?

---

### Activity 3

1. Use the same figure from Activity 2. Drag points around the circle so that the intersection $G$ is now outside the circle. Move $H$ so it is between $E$ and $D$ and $I$ is between $C$ and $F$, as shown.

2. Measure $\angle FGC$ formed by the secant lines and measure $\overarc{CIF}$ and $\overarc{DHE}$.

3. Drag points around the circle and examine the changes in measures. Fill in the angle and arc measures in a chart like the one below. Can you make a conjecture about the relationship between the angle measure and the two arc measures?

<table>
<thead>
<tr>
<th>$m\angle FGC$</th>
<th>$m\overarc{CIF}$</th>
<th>$m\overarc{DHE}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

### Fill in the angle and arc measures

Fill in the angle and arc measures in a chart like the one below. Try to create acute, right, and obtuse angles. Can you make a conjecture about the relationship between the angle measure and the two arc measures?

<table>
<thead>
<tr>
<th>$m\angle DGF$</th>
<th>$m\overarc{CHE}$</th>
<th>$m\overarc{DIF}$</th>
<th>Sum of Arcs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

### Try This

5. Fill in the angle and arc measures in a chart like the one below. Try to create acute, right, and obtuse angles. Can you make a conjecture about the relationship between the angle measure and the two arc measures?
Objectives
Find the measures of angles formed by lines that intersect circles.
Use angle measures to solve problems.

Who uses this?
Circles and angles help optometrists correct vision problems. (See Example 4.)

Theorem 11-5-1 connects arc measures and the measures of tangent-secant angles with tangent-chord angles.

Theorem 11-5-1

<table>
<thead>
<tr>
<th>THEOREM</th>
<th>HYPOTHESIS</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>If a tangent and a secant (or chord) intersect on a circle at the point of tangency, then the measure of the angle formed is half the measure of its intercepted arc.</td>
<td></td>
<td>$\angle ABC = \frac{1}{2}\overarc{AB}$</td>
</tr>
</tbody>
</table>

Tangent $\overline{BC}$ and secant $\overline{BA}$ intersect at $B$.

You will prove Theorem 11-5-1 in Exercise 45.

Example 1

Using Tangent-Secant and Tangent-Chord Angles

Find each measure.

A $\angle BCD$

$m\angle BCD = \frac{1}{2}m\overarc{BC}$

$m\angle BCD = \frac{1}{2}(142^\circ)$

$m\angle BCD = 71^\circ$

B $\overarc{ABC}$

$m\angle ACD = \frac{1}{2}m\overarc{ABC}$

$90^\circ = \frac{1}{2}m\overarc{ABC}$

$180^\circ = m\overarc{ABC}$

Find each measure.

1a. $\angle STU$

1b. $\angle SRU$
**Theorem 11-5-2**

If two secants or chords intersect in the interior of a circle, then the measure of each angle formed is half the sum of the measures of its intercepted arcs.

**PROOF**

Given: $\overline{AD}$ and $\overline{BC}$ intersect at $E$.

Prove: $m\angle 1 = \frac{1}{2}(m\overarc{AB} + m\overarc{CD})$

Proof:

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\overline{AD}$ and $\overline{BC}$ intersect at $E$.</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. Draw $\overline{BD}$.</td>
<td>2. Two pts. determine a line.</td>
</tr>
<tr>
<td>3. $m\angle 1 = m\angle EDB + m\angle EBD$</td>
<td>3. Ext. $\angle$ Thm.</td>
</tr>
<tr>
<td>4. $m\angle EDB = \frac{1}{2} m\overarc{AB}$, $m\angle EBD = \frac{1}{2} m\overarc{CD}$</td>
<td>4. Inscribed $\angle$ Thm.</td>
</tr>
<tr>
<td>5. $m\angle 1 = \frac{1}{2} m\overarc{AB} + \frac{1}{2} m\overarc{CD}$</td>
<td>5. Subst.</td>
</tr>
<tr>
<td>6. $m\angle 1 = \frac{1}{2} (m\overarc{AB} + m\overarc{CD})$</td>
<td>6. Distrib. Prop.</td>
</tr>
</tbody>
</table>

**EXAMPLE 2**

Finding Angle Measures Inside a Circle

Find each angle measure.

$m\angle SQR$

$m\angle SQR = \frac{1}{2} (m\overarc{PT} + m\overarc{SR})$

$= \frac{1}{2} (32^\circ + 100^\circ)$

$= \frac{1}{2} (132^\circ)$

$= 66^\circ$

**CHECK IT OUT!**

Find each angle measure.

2a. $m\angle ABD$

2b. $m\angle RNQ$
**Theorem 11-5-3**

If a **tangent and a secant**, **two tangents**, or **two secants** intersect in the exterior of a circle, then the measure of the angle formed is half the difference of the measures of its intercepted arcs.

\[ m\angle 1 = \frac{1}{2}(m\overarc{AD} - m\overarc{BD}) \quad m\angle 2 = \frac{1}{2}(m\overarc{EHG} - m\overarc{EG}) \quad m\angle 3 = \frac{1}{2}(m\overarc{N} - m\overarc{KM}) \]

You will prove Theorem 11-5-3 in Exercises 34–36.

---

### Example 3

**Finding Measures Using Tangents and Secants**

Find the value of \( x \).

**A**

\[ x = \frac{1}{2}(m\overarc{PS} - m\overarc{QS}) \]
\[ = \frac{1}{2}(174^\circ - 98^\circ) \]
\[ = 38^\circ \]

**B**

\[ x = \frac{1}{2}(m\overarc{EHG} - m\overarc{EG}) \]
\[ = \frac{1}{2}(228^\circ - 132^\circ) \]
\[ = 48^\circ \]

3. Find the value of \( x \).

**Example 4**

**Biology Application**

When a person is farsighted, light rays enter the eye and are focused behind the retina. In the eye shown, light rays converge at \( R \). If \( m\overarc{PS} = 60^\circ \) and \( m\overarc{QT} = 14^\circ \), what is \( m\angle PRS \)?

\[ m\angle PRS = \frac{1}{2}(m\overarc{PS} - m\overarc{QT}) \]
\[ = \frac{1}{2}(60^\circ - 14^\circ) \]
\[ = \frac{1}{2}(46^\circ) = 23^\circ \]

4. Two of the six muscles that control eye movement are attached to the eyeball and intersect behind the eye. If \( m\overarc{AEB} = 225^\circ \), what is \( m\angle ACB \)?
**Angle Relationships in Circles**

**Vertex of the Angle** | **Measure of Angle** | **Diagrams**
--- | --- | ---
**On a Circle** | Half the measure of its intercepted arc | ![Diagram](image1.png)

\[ m\angle 1 = 60° \quad m\angle 2 = 100° \]

**Inside a Circle** | Half the sum of the measures of its intercepted arcs | ![Diagram](image2.png)

\[ m\angle 1 = \frac{1}{2} (44° + 86°) = 65° \]

**Outside a Circle** | Half the difference of the measures of its intercepted arcs | ![Diagram](image3.png)

\[ m\angle 1 = \frac{1}{2} (202° - 78°) = 62° \]
\[ m\angle 2 = \frac{1}{2} (125° - 45°) = 40° \]

---

**Example 5**

**Finding Arc Measures**

Find \( \overparen{AF} \).

**Step 1** Find \( \overparen{ADB} \).

\[ m\angle ABC = \frac{1}{2} m\overparen{ADB} \]

If a tangent and secant intersect on a \( \odot \) at the pt. of tangency, then the measure of the \( \angle \) formed is half the measure of its intercepted arc.

\[ 110° = \frac{1}{2} m\overparen{ADB} \]

\[ m\overparen{ADB} = 220° \]

Substitute 110 for \( m\angle ABC \).

Multiply both sides by 2.

**Step 2** Find \( m\overparen{AD} \).

\[ m\overparen{ADB} = m\overparen{AD} + m\overparen{DB} \quad \text{Arc Add. Post.} \]

\[ 220° = m\overparen{AD} + 160° \quad \text{Substitute.} \]

\[ m\overparen{AD} = 60° \quad \text{Subtract 160 from both sides.} \]

**Step 3** Find \( m\overparen{AF} \).

\[ m\overparen{AF} = 360° - (m\overparen{AD} + m\overparen{DB} + m\overparen{BF}) \quad \text{Def. of a} \odot \]

\[ = 360° - (60° + 160° + 48°) \quad \text{Substitute.} \]

\[ = 92° \quad \text{Simplify.} \]

---

5. Find \( m\overparen{LP} \).
THINK AND DISCUSS

1. Explain how the measure of an angle formed by two chords of a circle is related to the measure of the angle formed by two secants.

2. GET ORGANIZED Copy and complete the graphic organizer. In each box write a theorem and draw a diagram according to where the angle’s vertex is in relationship to the circle.

GUIDED PRACTICE

Find each measure.

1. \( m \angle DAB \)
2. \( m \overarc{AC} \)
3. \( m \overarc{PN} \)
4. \( m \angle MNP \)
5. \( m \angle STU \)
6. \( m \angle HFG \)
7. \( m \angle NPK \)

Find the value of \( x \).

8. \( x \)
9. \( x \)
10. \( x \)

11. Science A satellite orbits Mars. When it reaches \( S \) it is about 12,000 km above the planet. How many arc degrees of the planet are visible to a camera in the satellite?
**Multi-Step** Find each measure.

12. \( m\widehat{DF} \)

13. \( m\widehat{CD} \)

14. \( m\widehat{PN} \)

15. \( m\widehat{KN} \)

---

**Independent Practice**

Find each measure.

16. \( m\angle BCD \)

17. \( m\angle ABC \)

18. \( m\angle XZW \)

19. \( m\angle XZV \)

20. \( m\angle QPR \)

21. \( m\angle ABC \)

22. \( m\angle MKJ \)

Find the value of \( x \).

23.

24.

25.

26. **Archaeology** Stonehenge is a circular arrangement of massive stones near Salisbury, England. A viewer at \( V \) observes the monument from a point where two of the stones \( A \) and \( B \) are aligned with stones at the endpoints of a diameter of the circular shape. Given that \( m\widehat{AB} = 48^\circ \), what is \( m\angle AVB? \)

**Multi-Step** Find each measure.

27. \( m\widehat{EG} \)

28. \( m\widehat{DE} \)

29. \( m\widehat{PR} \)

30. \( m\widehat{LP} \)
In the diagram, \( m \angle ABC = x^\circ \). Write an expression in terms of \( x \) for each of the following.
31. \( \widehat{AB} \)  
32. \( m \angle ABD \)  
33. \( \widehat{AEB} \)

34. Given: Tangent \( \overrightarrow{CD} \) and secant \( \overrightarrow{CA} \)  
Prove: \( m \angle ACD = \frac{1}{2}(m \widehat{AD} - m \widehat{BD}) \)  
Plan: Draw auxiliary line segment \( \overline{BD} \). Use the Exterior Angle Theorem to show that \( m \angle ACD = m \angle ABD - m \angle BDC \). Then use the Inscribed Angle Theorem and Theorem 11-5-1.

35. Given: Tangents \( \overrightarrow{FE} \) and \( \overrightarrow{FG} \)  
Prove: \( m \angle EFG = \frac{1}{2}(m \widehat{EHG} - m \widehat{EG}) \)

36. Given: Secants \( \overline{LJ} \) and \( \overline{LN} \)  
Prove: \( m \angle JLN = \frac{1}{2}(m \widehat{LN} - m \widehat{KM}) \)

37. Critical Thinking Suppose two secants intersect in the exterior of a circle as shown. What is greater, \( m \angle 1 \) or \( m \angle 2 \)? Justify your answer.

38. Write About It The diagrams show the intersection of perpendicular lines on a circle, inside a circle, and outside a circle. Explain how you can use these to help you remember how to calculate the measures of the angles formed.

39. Algebra Find the measures of the three angles of \( \triangle ABC \).

40. \( (2x - 10)^\circ \)  
\( 160^\circ \)

41. This problem will prepare you for the Multi-Step Test Prep on page 806.
The design was made by placing six equally-spaced points on a circle and connecting them.

a. Find \( m \angle BHC \).

b. Find \( m \angle EGD \).

c. Classify \( \triangle EGD \) by its angle measures and by its side lengths.
42. What is \( m\angle DCE \)?
   - A 19°
   - B 21°
   - C 79°
   - D 101°

43. Which expression can be used to calculate \( m\angle ABC \)?
   - F \( \frac{1}{2}(m\overarc AD + m\overarc AF) \)
   - H \( \frac{1}{2}(m\overarc DE - m\overarc AF) \)
   - G \( \frac{1}{2}(m\overarc DE + m\overarc AF) \)
   - J \( \frac{1}{2}(m\overarc AD - m\overarc AF) \)

44. **Gridded Response** In \( \odot Q \), \( m\overarc MN = 146° \) and \( m\angle JLK = 45° \). Find the degree measure of \( \overarc JK \).

**CHALLENGE AND EXTEND**

45. Prove Theorem 11-5-1.
   - Given: Tangent \( BC \) and secant \( BA \)
   - Prove: \( m\angle ABC = \frac{1}{2}m\overarc AB \)
   - (Hint: Consider two cases, one where \( AB \) is a diameter and one where \( AB \) is not a diameter.)

46. Given: \( \overline{YZ} \) and \( \overline{WZ} \) are tangent to \( \odot X \). \( m\overarc WY = 90° \)
   - Prove: \( WXYZ \) is a square.

47. Find \( x \).

48. Find \( m\overarc GH \).

**SPIRAL REVIEW**

Determine whether the ordered pair \((7, -8)\) is a solution of the following functions.
(Previous course)

49. \( g(x) = 2x^2 - 15x - 1 \)
50. \( f(x) = 29 - 3x \)
51. \( y = -\frac{7}{8}x \)

Find the volume of each pyramid or cone. Round to the nearest tenth. (Lesson 10-7)

52. regular hexagonal pyramid with a base edge of 4 m and a height of 7 m
53. right cone with a diameter of 12 cm and lateral area of \( 60\pi \text{ cm}^2 \)
54. regular square pyramid with a base edge of 24 in. and a surface area of 1200 \( \text{ in}^2 \)

In \( \odot P \), find each angle measure. (Lesson 11-4)

55. \( m\angle BCA \)
56. \( m\angle DBC \)
57. \( m\angle ADC \)

---

11-5 Angle Relationships in Circles 789
Explore Segment Relationships in Circles

When secants, chords, or tangents of circles intersect, they create several segments. You will measure these segments and investigate their relationships.

**Activity 1**

1. Construct a circle with center A. Label the point on the circle as B. Construct two secants CD and EF that intersect outside the circle at G. Hide B if desired. (It controls the circle’s size.)

2. Measure GC, GD, GE, and GF. Drag points around the circle and examine the changes in the measurements.

3. Fill in the segment lengths in a chart like the one below. Find the products of the lengths of segments on the same secant. Can you make a conjecture about the relationship of the segments formed by intersecting secants of a circle?

<table>
<thead>
<tr>
<th>GC</th>
<th>GD</th>
<th>GC • GD</th>
<th>GE</th>
<th>GF</th>
<th>GE • GF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Try This**

1. Make a sketch of the diagram from Activity 1, and create CF and DE to create ΔCFG and ΔEDG as shown.

2. Name pairs of congruent angles in the diagram. How are ΔCFG and ΔEDG related? Explain your reasoning.

3. Write a proportion involving sides of the triangles. Cross-multiply and state the result. What do you notice?

**Activity 2**

1. Construct a new circle with center A. Label the point on the circle as B. Create a radius segment from A to a new point C on the circle.

2. Construct a line through C perpendicular to radius AC. Create a new point D on this line, which is tangent to circle A at C. Hide radius AC.
Activity 3

3. Create a secant line through $D$ that intersects the circle at two new points $E$ and $F$, as shown.

4. Measure $DC$, $DE$, and $DF$. Drag points around the circle and examine the changes in the measurements. Fill in the measurements in a chart like the one below. Can you make a conjecture about the relationship between the segments of a tangent and a secant of a circle?

<table>
<thead>
<tr>
<th>$DE$</th>
<th>$DF$</th>
<th>$DE \cdot DF$</th>
<th>$DC$</th>
<th>$?$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Try This

4. How are the products for a tangent and a secant similar to the products for secant segments?

5. Try dragging $E$ and $F$ so they overlap (to make the secant segment look like a tangent segment). What do you notice about the segment lengths you measured in Activity 2? Can you state a relationship about two tangent segments from the same exterior point?

6. Challenge Write a formal proof of the relationship you found in Problem 2.

Activity 3

1. Construct a new circle with two chords $CD$ and $EF$ that intersect inside the circle at $G$.

2. Measure $GC$, $GD$, $GE$, and $GF$. Drag points around the circle and examine the changes in the measurements.

3. Fill in the segment lengths in a chart like the ones used in Activities 1 and 2. Find the products of the lengths of segments on the same chord. Can you make a conjecture about the relationship of the segments formed by intersecting chords of a circle?

Try This

7. Connect the endpoints of the chords to form two triangles. Name pairs of congruent angles. How are the two triangles that are formed related? Explain your reasoning.

8. Examine the conclusions you made in all three activities about segments formed by secants, chords, and tangents in a circle. Summarize your findings.
Objectives
Find the lengths of segments formed by lines that intersect circles.
Use the lengths of segments in circles to solve problems.

Vocabulary
secant segment
external secant segment
tangent segment

Who uses this?
Archaeologists use facts about segments in circles to help them understand ancient objects. (See Example 2.)

In 1901, divers near the Greek island of Antikythera discovered several fragments of ancient items. Using the mathematics of circles, scientists were able to calculate the diameters of the complete disks.

The following theorem describes the relationship among the four segments that are formed when two chords intersect in the interior of a circle.

**Theorem 11-6-1** Chord-Chord Product Theorem

If two chords intersect in the interior of a circle, then the products of the lengths of the segments of the chords are equal.

\[ AE \cdot EB = CE \cdot ED \]

Chords \( \overline{AB} \) and \( \overline{CD} \) intersect at \( E \).

You will prove Theorem 11-6-1 in Exercise 28.

**Example 1**

**Applying the Chord-Chord Product Theorem**

Find the value of \( x \) and the length of each chord.

\[
\begin{align*}
PQ \cdot QR &= SQ \cdot QT \\
6(4) &= x(8) \\
24 &= 8x \\
3 &= x
\end{align*}
\]

\[
\begin{align*}
PR &= 6 + 4 = 10 \\
ST &= 3 + 8 = 11
\end{align*}
\]

1. Find the value of \( x \) and the length of each chord.
**Example 2** Archaeology Application

Archaeologists discovered a fragment of an ancient disk. To calculate its original diameter, they drew a chord $AB$ and its perpendicular bisector $PQ$. Find the disk's diameter.

Since $PQ$ is the perpendicular bisector of a chord, $PR$ is a diameter of the disk.

\[
AQ \cdot QB = PQ \cdot QR
\]

\[
5(5) = 3(QR)
\]

\[
25 = 3QR
\]

\[
QR = \frac{25}{3} \text{ in.}
\]

\[
PR = 3 + \frac{25}{3} = \frac{3}{3} + \frac{25}{3} = \frac{28}{3} \text{ in.}
\]

2. **What if...?** Suppose the length of chord $AB$ that the archaeologists drew was 12 in. In this case how much longer is the disk's diameter compared to the disk in Example 2?

A **secant segment** is a segment of a secant with at least one endpoint on the circle. An **external secant segment** is a secant segment that lies in the exterior of the circle with one endpoint on the circle.

![Diagram showing secant and external secant segments]

**Theorem 11-6-2** Secant-Secant Product Theorem

<table>
<thead>
<tr>
<th>THEOREM</th>
<th>HYPOTHESIS</th>
<th>CONCLUSION</th>
</tr>
</thead>
<tbody>
<tr>
<td>If two secants intersect in the exterior of a circle, then the product of the lengths of one secant segment and its external segment equals the product of the lengths of the other secant segment and its external segment. (whole · outside = whole · outside)</td>
<td></td>
<td>$AE \cdot BE = CE \cdot DE$</td>
</tr>
</tbody>
</table>

**Proof**

Given: Secant segments $\overline{AE}$ and $\overline{CE}$

Prove: $AE \cdot BE = CE \cdot DE$

Proof: Draw auxiliary line segments $\overline{AD}$ and $\overline{CB}$. $\angle EAD$ and $\angle ECB$ both intercept $BD$, so $\angle EAD \cong \angle ECB$. $\angle E \cong \angle E$ by the Reflexive Property of $\cong$.

Thus $\triangle EAD \sim \triangle ECB$ by AA Similarity. Therefore corresponding sides are proportional, and $\frac{AE}{CE} = \frac{DE}{BE}$. By the Cross Products Property, $AE \cdot BE = CE \cdot DE$. 

![Diagram illustrating the proof of the Secant-Secant Product Theorem]
**Example 3**

Applying the Secant-Secant Product Theorem

Find the value of $x$ and the length of each secant segment.

\[ RT \cdot RS = RQ \cdot RP \]

\[ 10(4) = (x + 5)5 \]

\[ 40 = 5x + 25 \]

\[ 15 = 5x \]

\[ 3 = x \]

\[ RT = 4 + 6 = 10 \]

\[ RQ = 5 + 3 = 8 \]

---

3. Find the value of $z$ and the length of each secant segment.

A **tangent segment** is a segment of a tangent with one endpoint on the circle. $\overline{AB}$ and $\overline{AC}$ are tangent segments.

---

**Theorem 11-6-3**

Secant-Tangent Product Theorem

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Hypothesis</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Secant $\overline{AC}$ and tangent $\overline{DC}$ intersect at $C$.</td>
<td>$AC \cdot BC = DC^2$</td>
<td></td>
</tr>
</tbody>
</table>

You will prove Theorem 11-6-3 in Exercise 29.

---

**Example 4**

Applying the Secant-Tangent Product Theorem

Find the value of $x$.

\[ SQ \cdot RQ = PQ^2 \]

\[ 9(4) = x^2 \]

\[ 36 = x^2 \]

\[ \pm 6 = x \]

The value of $x$ must be 6 since it represents a length.

4. Find the value of $y$. 
THINK AND DISCUSS

1. Does the Chord-Chord Product Theorem apply when both chords are diameters? If so, what does the theorem tell you in this case?
2. Given A in the exterior of a circle, how many different tangent segments can you draw with A as an endpoint?
3. GET ORGANIZED Copy and complete the graphic organizer.

<table>
<thead>
<tr>
<th>Theorem</th>
<th>Diagram</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chord-Chord</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Secant-Secant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Secant-Tangent</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

GUIDED PRACTICE

1. **Vocabulary** \( \overline{AB} \) intersects \( \odot P \) at exactly one point. Point A is in the exterior of \( \odot P \), and point B lies on \( \odot P \). \( \overline{AB} \) is a(n) ? (tangent segment or external secant segment)

Find the value of the variable and the length of each chord.

2. \[ \begin{align*}
4 & = K + 8 \\
3 & = y + 8 \\
6 & = y + 3 \\
8 & = y + x \\
10 & = x \\
\end{align*} \]

3. \[ \begin{align*}
x & = 6 \\
x & = 4 \\
y & = 6 \\
\end{align*} \]

4. \[ \begin{align*}
6 & = y + 8 \\
10 & = y + 8 \\
\end{align*} \]

5. **Engineering** A section of an aqueduct is based on an arc of a circle as shown. \( EF \) is the perpendicular bisector of \( GH \). \( GH = 50 \text{ ft} \), and \( EF = 20 \text{ ft} \). What is the diameter of the circle?

6. \[ \begin{align*}
x & = 7.2 \\
x & = 7.2 \\
x & = 9 \\
\end{align*} \]

7. \[ \begin{align*}
y & = 7 \\
\end{align*} \]

8. \[ \begin{align*}
x & = 10.7 \\
x & = 11.5 \\
\end{align*} \]
Find the value of the variable.

9. \[ \triangle ABC \]

10. \[ \triangle MNP \]

11. \[ \triangle UTR \]

PRACTICE AND PROBLEM SOLVING

Find the value of the variable and the length of each chord.

12. \[ \triangle DEF \]

13. \[ \triangle JKM \]

14. \[ \triangle UYV \]

15. **Geology** Molokini is a small, crescent-shaped island \( \frac{2\frac{1}{2}}{2} \) miles from the Maui coast. It is all that remains of an extinct volcano. To approximate the diameter of the mouth of the volcano, a geologist can use a diagram like the one shown. What is the approximate diameter of the volcano’s mouth to the nearest foot?

Find the value of the variable and the length of each secant segment.

16. \[ \triangle ABC \]

17. \[ \triangle JKL \]

18. \[ \triangle OQP \]

Find the value of the variable.

19. \[ \triangle WUV \]

20. \[ \triangle ABD \]

21. \[ \triangle EFG \]

Use the diagram for Exercises 22 and 23.

22. \( M \) is the midpoint of \( \overline{PQ} \). \( RM = 10 \text{ cm} \), and \( PQ = 24 \text{ cm} \).
   a. Find \( MS \).
   b. Find the diameter of \( \odot O \).

23. \( M \) is the midpoint of \( \overline{PQ} \). The diameter of \( \odot O \) is 13 in., and \( RM = 4 \text{ in} \).
   a. Find \( PM \).
   b. Find \( PQ \).
Multi-Step  Find the value of both variables in each figure.

24. 

25. 

26. **Meteorology** A weather satellite $S$ orbits Earth at a distance $SE$ of 6000 mi. Given that the diameter of the earth is approximately 8000 mi, what is the distance from the satellite to $P$? Round to the nearest mile.

27. **ERROR ANALYSIS** The two solutions show how to find the value of $x$. Which solution is incorrect? Explain the error.

28. Prove Theorem 11-6-1.
Given: Chords $\overline{AB}$ and $\overline{CD}$ intersect at point $E$.
Prove: $\overline{AE} \cdot \overline{EB} = \overline{CE} \cdot \overline{ED}$

Plan: Draw auxiliary line segments $\overline{AC}$ and $\overline{BD}$. Show that $\triangle ECA \sim \triangle EBD$. Then write a proportion comparing the lengths of corresponding sides.

29. Prove Theorem 11-6-3.
Given: Secant segment $\overline{AC}$, tangent segment $\overline{DC}$
Prove: $\overline{AC} \cdot \overline{BC} = \overline{DC}^2$

30. **Critical Thinking** A student drew a circle and two secant segments. By measuring with a ruler, he found $\overline{PQ} \cong \overline{PS}$. He concluded that $\overline{QR} \cong \overline{ST}$. Do you agree with the student’s conclusion? Why or why not?

31. **Write About It** The radius of $\odot A$ is 4. $\overline{CD} = 4$, and $\overline{CB}$ is a tangent segment. Describe two different methods you can use to find $\overline{BC}$.

32. This problem will prepare you for the Multi-Step Test Prep on page 806.
Some Native American designs are based on eight points that are placed around the circumference of a circle. In $\odot O$, $\overline{BE} = 3$ cm, $\overline{AE} = 5.2$ cm, and $\overline{EC} = 4$ cm.

a. Find $\overline{DE}$ to the nearest tenth.

b. What is the diameter of the circle to the nearest tenth?

c. What is the length of $\overline{OE}$ to the nearest hundredth?
33. Which of these is closest to the length of tangent \( \overline{PQ} \)?

- A. 6.9
- B. 9.2
- C. 9.9
- D. 10.6

34. What is the length of \( \overline{UT} \)?

- F. 5
- G. 7
- H. 12
- I. 14

35. **Short Response** In \( \odot A \), \( \overline{AB} \) is the perpendicular bisector of \( \overline{CD} \). \( CD = 12 \), and \( EB = 3 \). Find the radius of \( \odot A \). Explain your steps.

### CHALLENGE AND EXTEND

36. **Algebra** \( \overline{KL} \) is a tangent segment of \( \odot N \).

   a. Find the value of \( x \).
   
   b. Classify \( \triangle KLM \) by its angle measures. Explain.

37. \( \overline{PQ} \) is a tangent segment of a circle with radius 4 in. \( Q \) lies on the circle, and \( PQ = 6 \) in. Find the distance from \( P \) to the circle. Round to the nearest tenth of an inch.

38. The circle in the diagram has radius \( c \). Use this diagram and the Chord-Chord Product Theorem to prove the Pythagorean Theorem.

39. Find the value of \( y \) to the nearest hundredth.

### SPIRAL REVIEW

40. An experiment was conducted to find the probability of rolling two threes in a row on a number cube. The probability was 3.5%. How many trials were performed in this experiment if 14 favorable outcomes occurred? (*Previous course*)

41. Two coins were flipped together 50 times. In 36 of the flips, at least one coin landed heads up. Based on this experiment, what is the experimental probability that at least one coin will land heads up when two coins are flipped? (*Previous course*)

Name each of the following. (*Lesson 1-1*)

42. two rays that do not intersect

43. the intersection of \( \overrightarrow{AC} \) and \( \overrightarrow{CD} \)

44. the intersection of \( \overrightarrow{CA} \) and \( \overrightarrow{BD} \)

Find each measure. Give your answer in terms of \( \pi \) and rounded to the nearest hundredth. (*Lesson 11-3*)

45. area of the sector \( XZW \)

46. arc length of \( \widehat{XW} \)

47. \( m \angle YZX \) if the area of the sector \( YZW \) is \( 40\pi \) ft\(^2\)

798  Chapter 11  Circles
Objectives
Write equations and graph circles in the coordinate plane. Use the equation and graph of a circle to solve problems.

Who uses this?
Meteorologists use circles and coordinates to plan the location of weather stations. (See Example 3.)

The equation of a circle is based on the Distance Formula and the fact that all points on a circle are equidistant from the center.

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]

Distance Formula

\[ r = \sqrt{(x - h)^2 + (y - k)^2} \]

Substitute the given values.

\[ r^2 = (x - h)^2 + (y - k)^2 \]

Square both sides.

Theorem 11-7-1 Equation of a Circle
The equation of a circle with center \((h, k)\) and radius \(r\) is \((x - h)^2 + (y - k)^2 = r^2\).

Example 1 Writing the Equation of a Circle
Write the equation of each circle.

A  \( \odot A \) with center \( A(4, -2) \) and radius 3

\[ (x - h)^2 + (y - k)^2 = r^2 \]

Equation of a circle

\[ (x - 4)^2 + (y - (-2))^2 = 3^2 \]

Substitute 4 for \( h \), -2 for \( k \), and 3 for \( r \). Simplify.

B  \( \odot B \) that passes through \((-2, 6)\) and has center \( B(-6, 3) \)

\[ r = \sqrt{(-2 - (-6))^2 + (6 - 3)^2} \]

Distance Formula

\[ = \sqrt{25} = 5 \]

Substitute -6 for \( h \), 3 for \( k \), and 5 for \( r \). Simplify.

(\( x + 6 \))^2 + (y - 3)^2 = 5^2

(\( x - (-6) \))^2 + (y - 3)^2 = 25

Write the equation of each circle.

1a. \( \odot P \) with center \( P(0, -3) \) and radius 8

1b. \( \odot Q \) that passes through \((2, 3)\) and has center \( Q(2, -1) \)
If you are given the equation of a circle, you can graph the circle by making a table or by identifying its center and radius.

**Example 2**

**Graphing a Circle**

Graph each equation.

**A** \( x^2 + y^2 = 25 \)

*Step 1* Make a table of values.

Since the radius is \( \sqrt{25} \), or 5, use \( \pm 5 \) and the values between for \( x \)-values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-5</th>
<th>-4</th>
<th>-3</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>( \pm 3 )</td>
<td>( \pm 4 )</td>
<td>( \pm 5 )</td>
<td>( \pm 4 )</td>
<td>( \pm 3 )</td>
<td>0</td>
</tr>
</tbody>
</table>

*Step 2* Plot the points and connect them to form a circle.

**B** \( (x + 1)^2 + (y - 2)^2 = 9 \)

The equation of the given circle can be written as \( (x - (-1))^2 + (y - 2)^2 = 3^2 \).

So \( h = -1 \), \( k = 2 \), and \( r = 3 \).

The center is \((-1, 2)\), and the radius is 3. Plot the point \((-1, 2)\). Then graph a circle having this center and radius 3.

**Check It Out!**

Graph each equation.

2a. \( x^2 + y^2 = 9 \)  
2b. \( (x - 3)^2 + (y + 2)^2 = 4 \)

**Student to Student**

**Graphing Circles**

I found a way to use my calculator to graph circles. You first need to write the circle’s equation in \( y = \) form.

For example, to graph \( x^2 + y^2 = 16 \), first solve for \( y \).

\[
\begin{align*}
y^2 &= 16 - x^2 \\
y &= \pm \sqrt{16 - x^2}
\end{align*}
\]

Now enter and graph the two equations

\[
y_1 = \sqrt{16 - x^2} \text{ and } y_2 = -\sqrt{16 - x^2}.
\]
Meteorology Application

Meteorologists are planning the location of a new weather station to cover Osceola, Waco, and Ireland, Texas. To optimize radar coverage, the station must be equidistant from the three cities which are located on a coordinate plane at \(A(2, 5)\), \(B(3, -2)\), and \(C(-5, -2)\).

a. What are the coordinates where the station should be built?

b. If each unit of the coordinate plane represents 8.5 miles, what is the diameter of the region covered by the radar?

Step 1 Plot the three given points.

Step 2 Connect \(A, B,\) and \(C\) to form a triangle.

Step 3 Find a point that is equidistant from the three points by constructing the perpendicular bisectors of two of the sides of \(\triangle ABC\).

The perpendicular bisectors of the sides of \(\triangle ABC\) intersect at a point that is equidistant from \(A, B,\) and \(C\).

The intersection of the perpendicular bisectors is \(P(-1, 1)\). \(P\) is the center of the circle that passes through \(A, B,\) and \(C\).

The weather station should be built at \(P(-1, 1)\), Clifton, Texas.

There are approximately 10 units across the circle. So the diameter of the region covered by the radar is approximately 85 miles.

3. **What if…?** Suppose the coordinates of the three cities in Example 3 are \(D(6, 2), E(5, -5),\) and \(F(-2, -4)\). What would be the location of the weather station?

**THINK AND DISCUSS**

1. What is the equation of a circle with radius \(r\) whose center is at the origin?

2. A circle has a diameter with endpoints \((1, 4)\) and \((-3, 4)\). Explain how you can find the equation of the circle.

3. Can a circle have a radius of \(-6\)? Justify your answer.

4. **GET ORGANIZED** Copy and complete the graphic organizer. First select values for a center and radius. Then use the center and radius you wrote to fill in the other circles. Write the corresponding equation and draw the corresponding graph.
Write the equation of each circle.
1. \(\odot A\) with center \(A(3, -5)\) and radius 12
2. \(\odot B\) with center \(B(-4, 0)\) and radius 7
3. \(\odot M\) that passes through \((2, 0)\) and that has center \(M(4, 0)\)
4. \(\odot N\) that passes through \((2, -2)\) and that has center \(N(-1, 2)\)

Multi-Step Graph each equation.
5. \((x - 3)^2 + (y - 3)^2 = 4\)
6. \((x - 1)^2 + (y + 2)^2 = 9\)
7. \((x + 3)^2 + (y + 4)^2 = 1\)
8. \((x - 3)^2 + (y + 4)^2 = 16\)
9. Communications A radio antenna tower is kept perpendicular to the ground by three wires of equal length. The wires touch the ground at three points on a circle whose center is at the base of the tower. The wires touch the ground at \(A(2, 6), B(-2, -2),\) and \(C(-5, 7)\).
   a. What are the coordinates of the base of the tower?
   b. Each unit of the coordinate plane represents 1 ft. What is the diameter of the circle?

Write the equation of each circle.
10. \(\odot R\) with center \(R(-12, -10)\) and radius 8
11. \(\odot S\) with center \(S(1.5, -2.5)\) and radius \(\sqrt{3}\)
12. \(\odot C\) that passes through \((2, 2)\) and that has center \(C(1, 1)\)
13. \(\odot D\) that passes through \((-5, 1)\) and that has center \(D(1, -2)\)

Multi-Step Graph each equation.
14. \(x^2 + (y - 2)^2 = 9\)
15. \((x + 1)^2 - y^2 = 16\)
16. \(x^2 + y^2 = 100\)
17. \(x^2 + (y + 2)^2 = 4\)
18. Anthropology Hundreds of stone circles can be found along the Gambia River in western Africa. The stones are believed to be over 1000 years old. In one of the circles at Ker Batch, three stones have approximate coordinates of \(A(3, 1), B(4, -2),\) and \(C(-6, -2)\).
   a. What are the coordinates of the center of the stone circle?
   b. Each unit of the coordinate plane represents 1 ft. What is the diameter of the stone circle?
Algebra  Write the equation of each circle.

19. \[ x^2 + y^2 = 1 \]

20. \[ x^2 + y^2 = 4 \]

21. Entertainment  In 2004, the world’s largest carousel was located at the House on the Rock, in Spring Green, Wisconsin. Suppose that the center of the carousel is at the origin and that one of the animals on the circumference of the carousel has coordinates \((24, 32)\).
   a. If one unit of the coordinate plane equals 1 ft, what is the diameter of the carousel?
   b. As the carousel turns, the animals follow a circular path. Write the equation of this circle.

Determine whether each statement is true or false. If false, explain why.

22. The circle \(x^2 + y^2 = 7\) has radius 7.

23. The circle \((x - 2)^2 + (y + 3)^2 = 9\) passes through the point \((-1, -3)\).

24. The center of the circle \((x - 6)^2 + (y + 4)^2 = 1\) lies in the second quadrant.

25. The circle \((x + 1)^2 + (y - 4)^2 = 4\) intersects the y-axis.

26. The equation of the circle centered at the origin with diameter 6 is \(x^2 + y^2 = 36\).

27. Estimation  You can use the graph of a circle to estimate its area.
   a. Estimate the area of the circle by counting the number of squares of the coordinate plane contained in its interior. Be sure to count partial squares.
   b. Find the radius of the circle. Then use the area formula to calculate the circle's area to the nearest tenth.
   c. Was your estimate in part a an overestimate or an underestimate?

28. Consider the circle whose equation is \((x - 4)^2 + (y + 6)^2 = 25\). Write, in point-slope form, the equation of the line tangent to the circle at \((1, -10)\).

29. This problem will prepare you for the Multi-Step Test Prep on page 806.

A hogan is a traditional Navajo home. An artist is using a coordinate plane to draw the symbol for a hogan. The symbol is based on eight equally spaced points placed around the circumference of a circle.
   a. She positions the symbol at \(A(-3, 5)\) and \(C(0, 2)\). What are the coordinates of \(E\) and \(G\)?
   b. What is the length of a diameter of the symbol?
   c. Use your answer from part b to write an equation of the circle.
Find the center and radius of each circle.
30. \((x - 2)^2 + (y + 3)^2 = 81\)  
31. \(x^2 + (y + 15)^2 = 25\)  
32. \((x + 1)^2 + y^2 = 7\)

Find the area and circumference of each circle. Express your answer in terms of \(\pi\).
33. circle with equation \((x + 2)^2 + (y - 7)^2 = 9\)
34. circle with equation \((x - 8)^2 + (y + 5)^2 = 7\)
35. circle with center \((-1, 3)\) that passes through \((2, -1)\)

36. **Critical Thinking** Describe the graph of the equation \(x^2 + y^2 = r^2\) when \(r = 0\).

37. **Geology** A seismograph measures ground motion during an earthquake. To find the epicenter of an earthquake, scientists take readings in three different locations. Then they draw a circle centered at each location. The radius of each circle is the distance the earthquake is from the seismograph. The intersection of the circles is the epicenter. Use the data below to find the epicenter of the New Madrid earthquake.

<table>
<thead>
<tr>
<th>Seismograph</th>
<th>Location</th>
<th>Distance to Earthquake</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>((-200, 200))</td>
<td>300 mi</td>
</tr>
<tr>
<td>B</td>
<td>((400, -100))</td>
<td>600 mi</td>
</tr>
<tr>
<td>C</td>
<td>((100, -500))</td>
<td>500 mi</td>
</tr>
</tbody>
</table>

38. For what value(s) of the constant \(k\) is the circle \(x^2 + (y - k)^2 = 25\) tangent to the \(x\)-axis?
39. \(\odot A\) has a diameter with endpoints \((-3, -2)\) and \((5, -2)\). Write the equation of \(\odot A\).
40. Recall that a locus is the set of points that satisfy a given condition. Draw and describe the locus of points that are 3 units from \((2, 2)\).
41. **Write About It** The equation of \(\odot P\) is \((x - 2)^2 + (y - 1)^2 = 9\). Without graphing, explain how you can determine whether the point \((3, -1)\) lies on \(\odot P\), in the interior of \(\odot P\), or in the exterior of \(\odot P\).

42. Which of these circles intersects the \(x\)-axis?
   - \(\text{A} \) \((x - 3)^2 + (y + 3)^2 = 4\)
   - \(\text{B} \) \((x + 1)^2 + (y - 4)^2 = 9\)
   - \(\text{C} \) \((x + 2)^2 + (y + 1)^2 = 1\)
   - \(\text{D} \) \((x + 1)^2 + (y + 4)^2 = 9\)
43. What is the equation of a circle with center \((-3, 5)\) that passes through the point \((1, 5)\)?
   - \(\text{F} \) \((x + 3)^2 + (y - 5)^2 = 4\)
   - \(\text{E} \) \((x + 3)^2 + (y - 5)^2 = 16\)
   - \(\text{G} \) \((x - 3)^2 + (y + 5)^2 = 4\)
   - \(\text{H} \) \((x - 3)^2 + (y + 5)^2 = 16\)
44. On a map of a park, statues are located at \((4, -2)\), \((-1, 3)\), and \((-5, -5)\). A circular path connects the three statues, and the circle has a fountain at its center. Find the coordinates of the fountain.
   - \(\text{A} \) \((-1, -2)\)
   - \(\text{B} \) \((2, 1)\)
   - \(\text{C} \) \((-2, 1)\)
   - \(\text{D} \) \((1, -2)\)
45. In three dimensions, the equation of a sphere is similar to that of a circle. The equation of a sphere with center \((h, j, k)\) and radius \(r\) is \((x - h)^2 + (y - j)^2 + (z - k)^2 = r^2\).
   a. Write the equation of a sphere with center \((2, -4, 3)\) that contains the point \((1, -2, -5)\).
   b. \(\overrightarrow{AC}\) and \(\overrightarrow{BC}\) are tangents from the same exterior point. If \(AC = 15\ m\), what is \(BC\)? Explain.

46. **Algebra** Find the point(s) of intersection of the line \(x + y = 5\) and the circle \(x^2 + y^2 = 25\) by solving the system of equations. Check your result by graphing the line and the circle.

47. Find the equation of the circle with center \((3, 4)\) that is tangent to the line whose equation is \(y = 2x + 3\). (Hint: First find the point of tangency.)

### Spiral Review

Simplify each expression. *(Previous course)*

48. \(\frac{2x^2 - 2(4x^2 + 1)}{2}\)

49. \(\frac{18a + 4(9a + 3)}{6}\)

50. \(3(x + 3y) - 4(3x + 2y) - (x - 2y)\)

In isosceles \(\triangle DEF\), \(\overline{DE} \cong \overline{EF}\). \(m\angle E = 60^\circ\), and \(m\angle D = (7x + 4)^\circ\). \(DE = 2y + 10\), and \(EF = 4y - 1\). Find the value of each variable. *(Lesson 4-8)*

51. \(x\)

52. \(y\)

Find each measure. *(Lesson 11-5)*

53. \(m\angle LNQ\)

54. \(m\angle NMP\)

---

**Career Path**

Bryan Moreno
Furniture Maker

Q: What math classes did you take in high school?
A: I took Algebra 1 and Geometry. I also took Drafting and Woodworking. Those classes aren’t considered math classes, but for me they were since math was used in them.

Q: What type of furniture do you make?
A: I mainly design and make household furniture, such as end tables, bedroom furniture, and entertainment centers.

Q: How do you use math?
A: Taking appropriate and precise measurements is very important. If wood is not measured correctly, the end result doesn’t turn out as expected. Understanding angle measures is also important. Some of the furniture I build has 30° or 40° angles at the edges.

Q: What are your future plans?
A: Someday I would love to design all the furniture in my own home. It would be incredibly satisfying to know that all my furniture was made with quality and attention to detail.
Angles and Segments in Circles

Native American Design
The members of a Native American cultural center are painting a circle of colors on their gallery floor. They start by laying out the circle and chords shown. Before they apply their paint to the design, they measure angles and lengths to check for accuracy.

1. The circle design is based on twelve equally spaced points placed around the circumference of the circle. As the group lays out the design, what should be \( m\angle AGB \)?

2. What should be \( m\angle KAE \)? Why?

3. What should be \( m\angle KMJ \)? Why?

4. The diameter of the circle is 22 ft. \( KM \approx 4.8 \text{ ft} \), and \( JM \approx 6.4 \text{ ft} \). What should be the length of \( MB \)?

5. The group members use a coordinate plane to help them position the design. Each square of a grid represents one square foot, and the center of the circle is at \((20, 14)\). What is the equation of the circle?

6. What are the coordinates of points \( L, C, F \), and \( I \)?
Quiz for Lessons 11-4 Through 11-7

11-4  Inscribed Angles
Find each measure.
1. \( \text{m} \angle BAC \)
2. \( \text{m} \overline{CD} \)

11-5  Angle Relationships in Circles
Find each measure.
5. \( \text{m} \angle RST \)
6. \( \text{m} \angle AEC \)

7. A manufacturing company is creating a plastic stand for DVDs. They want to make the stand with \( \text{m} \overline{MN} = 102^\circ \). What should be the measure of \( \angle MPN \)?

11-6  Segment Relationships in Circles
Find the value of the variable and the length of each chord or secant segment.
8. 
9. 

10. An archaeologist discovers a portion of a circular stone wall, shown by \( \overline{ST} \) in the figure. \( ST = 12.2 \text{ m} \) and \( UR = 3.9 \text{ m} \). What was the diameter of the original circular wall? Round to the nearest hundredth.

11-7  Circles in the Coordinate Plane
Write the equation of each circle.
11. \( \odot A \) with center \( A(-2, -3) \) and radius 3
12. \( \odot B \) that passes through \((1, 1)\) and that has center \( B(4, 5) \)
13. A television station serves residents of three cities located at \( J(5, 2), K(-7, 2) \), and \( L(-5, -8) \). The station wants to build a new broadcast facility that is equidistant from the three cities. What are the coordinates of the location where the facility should be built?
In a Cartesian coordinate system, a point is represented by the two coordinates \(x\) and \(y\). In a **polar coordinate system**, a point \(A\) is represented by its distance from the origin \(r\), and an angle \(\theta\), \(\theta\) is measured counterclockwise from the horizontal axis to \(OA\). The ordered pair \((r, \theta)\) represents the polar coordinates of point \(A\).

In a polar coordinate system, the origin is called the **pole**. The horizontal axis is called the **polar axis**.

You can use the equation of a circle \(r^2 = x^2 + y^2\) and the tangent ratio \(\theta = \frac{y}{x}\) to convert rectangular coordinates to polar coordinates.

### Example 1

**Converting Rectangular Coordinates to Polar Coordinates**

Convert \((3, 4)\) to polar coordinates.

\[
\begin{align*}
  r^2 & = x^2 + y^2 \\
  r & = 5 \\
  \tan \theta & = \frac{4}{3} \\
  \theta & = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ
\end{align*}
\]

The polar coordinates are \((5, 53^\circ)\).

1. Convert \((4, 1)\) to polar coordinates.

You can use the relationships \(x = r\cos \theta\) and \(y = r\sin \theta\) to convert polar coordinates to rectangular coordinates.

### Example 2

**Converting Polar Coordinates to Rectangular Coordinates**

Convert \((2, 130^\circ)\) to rectangular coordinates.

\[
\begin{align*}
  x & = r\cos \theta \\
  y & = r\sin \theta \\
  x & = 2\cos 130^\circ \\
  y & = 2\sin 130^\circ \\
  x & \approx -1.29 \\
  y & \approx 1.53
\end{align*}
\]

The rectangular coordinates are \((-1.29, 1.53)\).

2. Convert \((4, 60^\circ)\) to rectangular coordinates.
EXAMPLE 3

Plotting Polar Coordinates

Plot the point \((4, 225°)\).

Step 1 Measure 225° counterclockwise from the polar axis.

Step 2 Locate the point on the ray that is 4 units from the pole.

3. Plot the point \((4, 300°)\).

EXAMPLE 4

Graphing Polar Equations

Graph \(r = 4\).

Make a table of values and plot the points.

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>0°</th>
<th>45°</th>
<th>135°</th>
<th>270°</th>
<th>300°</th>
<th>360°</th>
</tr>
</thead>
<tbody>
<tr>
<td>(r)</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

4. Graph \(r = 2\).

EXTENSION

Exercises

Convert to polar coordinates.

1. \((2, 2)\)  2. \((1, 0)\)  3. \((3, 7)\)  4. \((0, 15)\)

Convert to rectangular coordinates.

5. \((3, 150°)\)  6. \((5, 214°)\)  7. \((4, 303°)\)  8. \((4.5, 90°)\)

Plot each point.

9. \((4, 45°)\)  10. \((3, 165°)\)  11. \((1, 240°)\)  12. \((3.5, 315°)\)

13. Critical Thinking Graph the equation \(r = 5\). What can you say about the graph of an equation of the form \(r = a\), where \(a\) is a positive real number?

Technology Graph each equation.

14. \(r = -5 \sin \theta\)  15. \(r = 3 \sin 4\theta\)  16. \(r = -4 \cos \theta\)

17. \(r = 5 \cos 3\theta\)  18. \(r = 3 \cos 2\theta\)  19. \(r = 2 + 4 \sin \theta\)
**Vocabulary**

adjacent arcs .................. 757    exterior of a circle ............. 746    secant segment .................. 793
arc ................................ 756    external secant segment ...... 793    sector of a circle ............... 764
arc length ...................... 766    intercepted arc .................. 772    segment of a circle .......... 765
central angle .................. 756    inscribed angle .................. 772    semicircle ....................... 756
chord .......................... 746    interior of a circle ............. 746    subtend .......................... 772
common tangent ............... 748    intercepted arc .................. 772    tangent of a circle .......... 746
concentric circles ............ 747    major arc ......................... 756    tangent circles ............... 747
congruent arcs ............... 757    minor arc ......................... 756    tangent segment ............ 794
congruent circles ............. 747    point of tangency ............... 746
intercepted arc ............... 772    secant .......................... 746
interior of a circle .......... 746    secant segment .................. 793
major arc ....................... 756    sector of a circle ............... 764
minor arc ....................... 756    semicircle ....................... 756
point of tangency ............. 746    tangent ........................... 746
secant .......................... 746    tangent circles ............... 747
tangent segment ............. 794

Complete the sentences below with vocabulary words from the list above.

1. A(n) ___?___ is a region bounded by an arc and a chord.

2. An angle whose vertex is at the center of a circle is called a(n) ___?___.

3. The measure of a(n) ___?___ is 360° minus the measure of its central angle.

4. ___?___ are coplanar circles with the same center.

**11-1 Lines That Intersect Circles** (pp. 746–754)

**EXAMPLES**

- Identify each line or segment that intersects \( \odot A \).

  chord: \( \overline{DE} \)
  tangent: \( \overline{BC} \)
  radii: \( \overline{AE}, \overline{AD}, \) and \( \overline{AB} \)
  secant: \( \overline{DE} \)
  diameter: \( \overline{DE} \)

- \( \overline{RS} \) and \( \overline{RW} \) are tangent to \( \odot T \). \( RS = x + 5 \) and \( RW = 3x - 7 \). Find \( RS \).

  \[
  RS = RW
  
  2 \text{ segs. tangent to } \odot \text{ from same ext. pt. } \implies \text{ segs. } \equiv.
  
  x + 5 = 3x - 7
  
  -2x + 5 = -7
  
  -2x = -12
  
  x = 6
  
  RS = 6 + 5
  
  = 11
  
  Substitute the given values.
  Subtract 3x from both sides.
  Subtract 5 from both sides.
  Divide both sides by \(-2\).
  Substitute 6 for \(y\).
  Simplify.

**EXERCISES**

Identify each line or segment that intersects each circle.

5. \( \overline{DE} \)

6. \( \overline{BC} \)

Given the measures of the following segments that are tangent to a circle, find each length.

7. \( AB = 9x - 2 \) and \( BC = 7x + 4 \). Find \( AB \).

8. \( EF = 5y + 32 \) and \( EG = 8 - y \). Find \( EG \).

9. \( JK = 8m - 5 \) and \( JL = 2m + 4 \). Find \( JK \).

10. \( WX = 0.8x + 1.2 \) and \( WY = 2.4x \). Find \( WY \).
11-2 Arcs and Chords (pp. 756–763)

**EXAMPLES**

Find each measure.
- \( \text{m} \overparen{BF} \)
  \( \angle BAF \) and \( \angle FAE \) are supplementary, so \( \text{m} \angle BAF = 180^\circ - 62^\circ = 118^\circ \).
  \( \text{m} \overparen{BF} = \text{m} \angle BAF = 118^\circ \)
- \( \text{m} \overparen{DF} \)
  Since \( \text{m} \angle DAE = 90^\circ \), \( \text{m} \overparen{DE} = 90^\circ \).
  \( \text{m} \angle EAF = 62^\circ \), so \( \text{m} \overparen{EF} = 62^\circ \).
  By the Arc Addition Postulate, \( \text{m} \overparen{DF} = \text{m} \overparen{DE} + \text{m} \overparen{EF} = 90^\circ + 62^\circ = 152^\circ \).

**EXERCISES**

Find each measure.
11. \( \text{m} \overparen{KM} \)
12. \( \text{m} \overparen{HK} \)
13. \( \text{m} \overparen{JK} \)
14. \( \text{m} \overparen{MJ} \)

Find each length to the nearest tenth.
15. \( \overline{ST} \)
16. \( \overline{CD} \)

---

11-3 Sector Area and Arc Length (pp. 764–769)

**EXAMPLES**

- Find the area of sector \( \overparen{PQR} \).
  Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.
  \[ A = \pi r^2 \left( \frac{\theta}{360^\circ} \right) \]
  \[ = \pi (4)^2 \left( \frac{135^\circ}{360^\circ} \right) \]
  \[ = 16\pi \left( \frac{3}{8} \right) \]
  \[ = 6\pi \text{ m}^2 \]
  \[ \approx 18.85 \text{ m}^2 \]
- Find the length of \( \overparen{AB} \).
  Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.
  \[ L = 2\pi r \left( \frac{\theta}{360^\circ} \right) \]
  \[ = 2\pi (9) \left( \frac{80^\circ}{360^\circ} \right) \]
  \[ = 18\pi \left( \frac{4}{9} \right) \]
  \[ = 8\pi \text{ ft} \]
  \[ \approx 25.13 \text{ ft} \]

**EXERCISES**

Find the area of each sector. Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.
17. \( \text{sector} \ \overparen{DEF} \)
18. \( \text{sector} \ \overparen{JKL} \)

Find each arc length. Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.
19. \( \overparen{GH} \)
20. \( \overparen{MN} \)
11-4 Inscribed Angles (pp. 772–779)

**Examples**

Find each measure.

- **m∠ABD**
  
  By the Inscribed Angle Theorem, m∠ABD = \( \frac{1}{2} \) m\(\widehat{AD}\), so m∠ABD = \( \frac{1}{2} \) (108°) = 54°.

- **m\(\overparen{BE}\)**
  
  By the Inscribed Angle Theorem, m∠BAE = \( \frac{1}{2} \) m\(\overparen{BE}\). So 28° = \( \frac{1}{2} \) m\(\overparen{BE}\), and m\(\overparen{BE}\) = 2(28°) = 56°.

**Exercises**

Find each measure.

21. m\(\overparen{L}\)

22. m∠MKL

Find each value.

23. x

24. m∠RSP

25. m\(\overparen{MR}\)

26. m∠QMR

27. m∠GKH

28. A piece of string art is made by placing 16 evenly spaced nails around the circumference of a circle. A piece of string is wound from A to B to C to D. What is m∠BXC?
**11-6 Segment Relationships in Circles (pp. 792–798)**

**EXAMPLES**

- Find the value of the variable and the length of each chord.
  
  \[ AE \cdot EB = DE \cdot EC \]
  
  \[ 12x = 8(6) \]
  
  \[ 12x = 48 \]
  
  \[ x = 4 \]
  
  \[ AB = 12 + 4 = 16 \]
  
  \[ DC = 8 + 6 = 14 \]

- Find the value of the variable and the length of each secant segment.
  
  \[ FJ \cdot FG = FK \cdot FH \]
  
  \[ 16(4) = (6 + x)6 \]
  
  \[ 64 = 36 + 6x \]
  
  \[ 28 = 6x \]
  
  \[ x = 4 \frac{2}{3} \]
  
  \[ FJ = 12 + 4 = 16 \]
  
  \[ FK = 4 \frac{2}{3} + 6 = 10 \frac{2}{3} \]

**EXERCISES**

Find the value of the variable and the length of each chord.

29.  

30.

Find the value of the variable and the length of each secant segment.

31.  

32.

**11-7 Circles in the Coordinate Plane (pp. 799–805)**

**EXAMPLES**

- Write the equation of \( \odot A \) that passes through \((-1, 1) \) and that has center \( A(2, 3) \).

  The equation of a circle with center \((h, k) \) and radius \( r \) is \((x - h)^2 + (y - k)^2 = r^2\).

  \[ r = \sqrt{(2 - (-1))^2 + (3 - 1)^2} = \sqrt{3^2 + 2^2} = \sqrt{13} \]

  The equation of \( \odot A \) is \((x - 2)^2 + (y - 3)^2 = 13\).

- Graph \((x - 2)^2 + (y + 1)^2 = 4\).

  The center of the circle is \((2, -1)\), and the radius is \( \sqrt{4} = 2\).

**EXERCISES**

Write the equation of each circle.

33.  \( \odot A \) with center \((-4, -3) \) and radius 3

34.  \( \odot B \) that passes through \((-2, -2) \) and that has center \( B(-2, 0) \)

35.  \( \odot C \)

36.  Graph \((x + 2)^2 + (y - 2)^2 = 1\).
1. Identify each line or segment that intersects the circle.

2. A jet is at a cruising altitude of 6.25 mi. To the nearest mile, what is the distance from the jet to a point on Earth's horizon? (Hint: The radius of Earth is 4000 mi.)

Find each measure.

3. \( \overline{JK} \)

4. \( UV \)

5. Find the area of the sector. Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.

6. Find the length of \( \overline{BC} \). Give your answer in terms of \( \pi \) and rounded to the nearest hundredth.

7. If \( m \angle SPR = 47^\circ \) in the diagram of a logo, find \( m \overline{SR} \).

8. A printer is making a large version of the logo for a banner. According to the specifications, \( m \overline{PQ} = 58^\circ \). What should the measure of \( \angle QTR \) be?

Find each measure.

9. \( m \angle ABC \)

10. \( m \angle NKL \)

11. A surveyor \( S \) is studying the positions of four columns \( A, B, C, \) and \( D \) that lie on a circle. He finds that \( m \angle CSD = 42^\circ \) and \( m \overline{CD} = 124^\circ \). What is \( m \overline{AB} \)?

Find the value of the variable and the length of each chord or secant segment.

12. 

13. 

14. The illustration shows a fragment of a circular plate. \( AB = 8 \text{ in.} \), and \( CD = 2 \text{ in.} \). What is the diameter of the plate?

15. Write the equation of the circle that passes through \((-2, 4)\) and that has center \((1, -2)\).

16. An artist uses a coordinate plane to plan a mural. The mural will include portraits of civic leaders at \( X(2, 4) \), \( Y(-6, 0) \), and \( Z(2, -8) \) and a circle that passes through all three portraits. What are the coordinates of the center of the circle?
FOCUS ON SAT MATHEMATICS SUBJECT TESTS

The topics covered on the SAT Mathematics Subject Tests vary only slightly each time the test is administered. You can find out the general distribution of questions across topics, then determine which areas need more of your attention when you are studying for the test.

You may want to time yourself as you take this practice test. It should take you about 6 minutes to complete.

1. $AC$ and $BD$ intersect at the center of the circle shown. If $m\angle BDC = 30^\circ$, what is the measure of minor $AB$?
   (A) 15°
   (B) 30°
   (C) 60°
   (D) 105°
   (E) 120°

   Note: Figure not drawn to scale.

2. Which of these is the equation of a circle that is tangent to the lines $x = 1$ and $y = 3$ and has radius 2?
   (A) $(x + 1)^2 + (y - 1)^2 = 4$
   (B) $(x - 1)^2 + (y + 1)^2 = 4$
   (C) $x^2 + (y - 1)^2 = 4$
   (D) $(x - 1)^2 + y^2 = 4$
   (E) $x^2 + y^2 = 4$

3. If $LK = 6$, $LN = 10$, and $PK = 3$, what is $PM$?
   (A) 7
   (B) 8
   (C) 9
   (D) 10
   (E) 11

4. Circle $D$ has radius 6, and $m\angle ABC = 25^\circ$. What is the length of minor $AC$?

   Note: Figure not drawn to scale.
   (A) $\frac{5\pi}{6}$
   (B) $\frac{5\pi}{4}$
   (C) $\frac{5\pi}{3}$
   (D) $3\pi$
   (E) $5\pi$

5. A square is inscribed in a circle as shown. If the radius of the circle is 9, what is the area of the shaded region, rounded to the nearest hundredth?
   (A) 11.56
   (B) 23.12
   (C) 57.84
   (D) 104.12
   (E) 156.23
**Multiple Choice: Choose Combinations of Answers**

Given a multiple-choice test item where you are asked to choose from a combination of statements, the correct response is the most complete answer choice available. A strategy to use when solving these types of test items is to compare each given statement with the question and determine if it is true or false. If you determine that more than one of the statements is correct, then you can choose the combination that contains each correct statement.

**Example 1**

Given that \( \ell \parallel m \) and \( n \) is a transversal, which statement(s) are correct?

I. \( \angle 1 \cong \angle 3 \)  
   II. \( \angle 2 \cong \angle 5 \)  
   III. \( \angle 2 \cong \angle 8 \)

- **A** I only  
- **B** I and II  
- **C** II only  
- **D** I and III

Look at each statement separately and determine if it is true or false. As you consider each statement, write true or false beside the statement.

Consider statement I: Because \( \angle 1 \) and \( \angle 3 \) are vertical angles and vertical angles are congruent, then this statement is TRUE. So the answer could be choice A, B, or D.

Consider statement II: \( \angle 2 \cong \angle 4 \) because they are vertical angles. \( \angle 4 \) and \( \angle 5 \) are supplementary angles because they are same-side interior angles. So \( \angle 2 \) and \( \angle 5 \) must be supplementary, not congruent. This statement is FALSE. The answer is NOT choice B or C.

Consider statement III: Because \( \angle 2 \) and \( \angle 8 \) are alternate exterior angles and alternate exterior angles are congruent, this statement is TRUE.

Since statements I and III are both true, choice D is correct.

You can also keep track of your statements in a table.

<table>
<thead>
<tr>
<th>Statement</th>
<th>True/False</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>TRUE</td>
</tr>
<tr>
<td>II</td>
<td>FALSE</td>
</tr>
<tr>
<td>III</td>
<td>TRUE</td>
</tr>
</tbody>
</table>

Only I and III are TRUE statements.
Read each test item and answer the questions that follow.

**Item A**
Which are chords of circle W?

I. \(\overline{AB}\)

II. \(\overline{WG}\)

III. \(\overline{EC}\)

IV. \(\overline{FD}\)

A  I only  C  I and II

B  III only  D  III and IV

1. What is the definition of a chord?
2. Determine if statements I, II, III, and IV are true or false. Explain your reasoning for each.
3. Kristin realized that statement III was true and selected choice B as her response. Do you agree? Why or why not?

**Item B**
Classify \(\triangle DEF\).

\(F\)  acute  \(H\)  obtuse

\(G\)  acute scalene  \(J\)  right equilateral

4. How can you use the Triangle Sum Theorem to find all of the angle measures of \(\triangle DEF\)?
5. Consider the angle measures of \(\triangle DEF\). Is the triangle acute, right, or obtuse?
6. Explain how you can use your answer to Problem 5 to eliminate two answer choices.
7. Can a triangle be classified in any other way than by its angles? Explain.
8. Which choice gives the most complete response?

**Item C**
Which describes the arc length of \(\widehat{AB}\)?

I. \(\frac{17}{72}(24\pi)\)

II. \(\frac{17\pi}{3}\)

III. \(\frac{17}{36}(24\pi)\)

A  I only  C  I and II

B  II only  D  I, II, and III

9. What is the formula to find arc length?
10. Is statement I true or false? Explain.
11. Decide if statement II is true or false. Should you select the answer choice yet? Why or why not?
13. Describe how you know which combination of statements is correct.

**Item D**
A rectangular prism has a length of 5 m, a height of 10 m, and a width of 4 m. Describe the change if the height and width of the prism are multiplied by \(\frac{1}{2}\).

I. The new volume is one fourth of the original volume.

II. The new height is 20 m, and the new width is 2 m.

III. The new surface area is less than half of the original surface area.

F  I only  H  I, II, and II

G  II and III  J  I and III

14. Create a table and determine if each statement is true or false.
15. Using your table, which choice is the most accurate?
Multiple Choice

1. The composite figure is a right prism that shares a base with the regular pentagonal pyramid on top. If the lateral area of this figure is 328 square feet, what is the slant height of the pyramid?

- 2.5 feet
- 8.4 feet
- 5.0 feet
- 9.0 feet

2. What is the area of the polygon with vertices A(2, 3), B(12, 3), C(6, 0), and D(2, 0)?

- 12 square units
- 30 square units
- 21 square units
- 42 square units

3. What is m\(\widehat{BC}\)?

- 36°
- 54°
- 45°
- 72°

4. If the length of \(\widehat{ED}\) is \(6\pi\) centimeters, what is the area of sector \(EFD\)?

- \(20\pi\) square centimeters
- \(72\pi\) square centimeters
- \(120\pi\) square centimeters
- \(240\pi\) square centimeters

5. Which of these line segments is NOT a chord of \(\odot F\)?

- \(\overline{EF}\)
- \(\overline{AF}\)
- \(\overline{CA}\)
- \(\overline{AE}\)

6. \(\triangle JKL\) is a right triangle where \(m\angle K = 90^\circ\) and \(\tan J = \frac{3}{4}\). Which of the following could be the side lengths of \(\triangle JKL\)?

- \(KL = 16, KJ = 12,\) and \(JL = 20\)
- \(KL = 15, KJ = 25,\) and \(JL = 20\)
- \(KL = 20, KJ = 16,\) and \(JL = 12\)
- \(KL = 18, KJ = 24,\) and \(JL = 30\)

Use the diagram for Items 7 and 8.

7. What is \(m\angle QU\)?

- 25°
- 58°
- 42°
- 71°

8. Which expression can be used to calculate the length of \(PS\)?

- \(PR \cdot PO\overline{PU}\)
- \(PO \cdot QR\overline{PU}\)
- \(PR \cdot PR\overline{PU}\)
- \(PO \cdot PR\overline{PS}\)

9. \(\triangle ABC\) has vertices A(0, 0), B(−1, 3), and C(2, 4). If \(\triangle ABC \sim \triangle DEF\) and \(\triangle DEF\) has vertices D(5, −3), E(4, −2), and F(3, \(y\)), what is the value of \(y\)?

- −7
- −3
- −5
- −1

10. What is the equation of the circle with diameter \(\overline{MN}\) that has endpoints M(−1, 1) and N(3, −5)?

- \((x + 1)^2 + (y − 2)^2 = 13\)
- \((x − 1)^2 + (y + 2)^2 = 13\)
- \((x + 1)^2 + (y − 2)^2 = 26\)
- \((x − 1)^2 + (y + 2)^2 = 52\)
11. Kite $PQRS$ has diagonals $PR$ and $QS$ that intersect at $T$. Which of the following is the shortest segment from $Q$ to $PR$?

- [ ] $PT$
- [ ] $PQ$
- [ ] $RQ$
- [ ] $TQ$

12. If the perimeter of an equilateral triangle is reduced by a factor of $\frac{1}{2}$, what is the effect on the area of the triangle?

- [ ] The area remains constant.
- [ ] The area is reduced by a factor of $\frac{1}{2}$.
- [ ] The area is reduced by a factor of $\frac{1}{4}$.
- [ ] The area is reduced by a factor of $\frac{1}{6}$.

13. The area of a right isosceles triangle is 36 m$^2$. What is the length of the hypotenuse of the triangle?

- [ ] 6 meters
- [ ] $6\sqrt{2}$ meters
- [ ] 12 meters
- [ ] $12\sqrt{2}$ meters

**Gridded Response**

14. The ratio of the side lengths of a triangle is 4:5:8. If the perimeter is 38.25 centimeters, what is the length in centimeters of the shortest side?

15. What is the geometric mean of 4 and 16?

16. For $\triangle HGI$ and $\triangle LMK$ suppose that $\angle H \equiv \angle L$, $HGI = 4x + 5$, $KL = 9$, $HJ = 5x - 1$, and $LM = 13$. What must be the value of $x$ to prove that $\triangle HGI$ and $\triangle LMK$ are congruent by SAS?

17. If the length of a side of a regular hexagon is 2, what is the area of the hexagon to the nearest tenth?

18. What is the arc length of a semicircle in a circle with radius 5 millimeters? Round to the nearest hundredth.

19. What is the surface area of a sphere whose volume is $288\pi$ cubic centimeters? Round to the nearest hundredth.

20. Convert $(6, 60^\circ)$ to rectangular coordinates. What is the value of the $x$-coordinate?

**Short Response**

21. Use the diagram to find the value of $x$. Show your work or explain in words how you determined your answer.

22. Paul needs to rent a storage unit. He finds one that has a length of 10 feet, a width of 5 feet, and a height of 9 feet. He finds a second storage unit that has a length of 11 feet, a width of 4 feet, and a height of 8 feet. Suppose that the first storage unit costs $85.00 per month and that the second storage unit costs $70.00 per month.

   a. Which storage unit has a lower price per cubic foot? Show your work or explain in words how you determined your answer.
   
   b. Paul finds a third storage unit that charges $0.25 per cubic foot per month. What are possible dimensions of the storage unit if the charge is $100.00 per month?

23. The equation of $\odot C$ is $x^2 + (y + 1)^2 = 25$.

   a. Graph $\odot C$.
   
   b. Write the equation of the line that is tangent to $\odot C$ at $(3, 3)$. Show your work or explain in words how you determined your answer.

24. A tangent and a secant intersect on a circle at the point of tangency and form an acute angle. Explain how you would find the range of possible measures for the intercepted arc.

**Extended Response**

25. Let $ABCD$ be a quadrilateral inscribed in a circle such that $\overline{AB} \parallel \overline{DC}$.

   a. Prove that $m\angle AD = m\angle BC$.
   
   b. Suppose $ABCD$ is a trapezoid. Show that $ABCD$ must be isosceles. Justify your answer.
   
   c. If $ABCD$ is not a trapezoid, explain why $ABCD$ must be a rectangle.